

Optimal first-order error estimates of a fully segregation scheme for the Navier-Stokes equations*

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Abstract

A first-order linear fully discrete scheme is studied for the incompressible time-dependent Navier-Stokes equations in three-dimensional domains. This scheme, based on an incremental pressure projection method, decouples each component of the velocity and the pressure, solving in each time step, a linear convection-diffusion problem for each component of the velocity and a Poisson-Neumann problem for the pressure.

Using first-order *inf-sup* stable C^0 -finite elements, optimal error estimates of order $O(k + h)$ are deduced without imposing constraints on h and k , the mesh size and the time step, respectively.

Finally, some numerical results are presented according the theoretical analysis, and also comparing to other current first-order segregated schemes.

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Introduction

Let us consider the Navier-Stokes system, associated to the dynamics of viscous and incompressible fluids filling a bounded domain $\Omega \subset \mathbb{R}^3$ in a time interval $(0, T)$:

$$(P) \quad \begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times (0, T), \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \times (0, T), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega. \end{cases}$$

where the unknowns are $\mathbf{u} : (\mathbf{x}, t) \in \Omega \times (0, T) \rightarrow \mathbb{R}^3$ the velocity field and $p : (\mathbf{x}, t) \in \Omega \times (0, T) \rightarrow \mathbb{R}$ the pressure, and data are $\nu > 0$ the viscosity coefficient (which is assumed constant for simplicity) and $\mathbf{f} : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ the external forces. We denote by ∇ the gradient operator and Δ the Laplace operator.

We consider a (uniform) partition of $[0, T]$ related to a fixed time step $k = T/M$: $t_0 = 0, t_1 = k, \dots, t_m = mk, \dots, t_M = T$. If $u = (u^m)_{m=0}^M$ is a given vector with $u^m \in X$ (a Banach space), let us to introduce the following notation for discrete in time norms:

$$\|u\|_{l^2(X)} = \left(k \sum_{m=0}^M \|u^m\|_X^2 \right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max_{m=0, \dots, M} \|u^m\|_X$$

For simplicity, we will denote $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0, T; H^1)$ etc., and $\mathbf{H}^1 = H^1(\Omega)^3$ etc. We will denote by $C > 0$ different constants, always independent of discrete parameters k and h .

The numerical analysis for the Navier-Stokes problem (P) has received much attention in the last decades and many numerical schemes are now available. The main (numerical) difficulties are: the coupling between the pressure term ∇p and the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ and the nonlinearity given by the convective terms $(\mathbf{u} \cdot \nabla) \mathbf{u}$.

Fractional-step projection methods are becoming widely used, splitting the different operators appearing in the problem. The origin of these methods is generally credited to the works of Chorin [4] and Temam [30]. They developed the well known *Chorin-Temam projection* method, which is a two-step scheme, computing firstly an intermediate velocity via a convection-diffusion problem and secondly a velocity-pressure pair via a divergence-free $L^2(\Omega)$ -projection problem. Afterwards, a modified projection scheme (called *incremental-pressure or Van-Kan scheme*) was developed [23], adding an explicit pressure term in the first step and a pressure correction term in the projection step. The main drawbacks of projection methods are that the end-of-step velocity does not satisfy the exact boundary conditions and the discrete pressure satisfies an “artificial” Neumann boundary condition.

Some current variants of projection methods are: rotational pressure-correction schemes ([33], [14], [15]), velocity-correction schemes ([11], [12]), consistent-splitting schemes ([13], [15],

[29]) and penalty pressure-projection schemes ([1], [2], [7]). Other variants can be seen in [24] and [25].

The convergence of the *Chorin-Temam projection* method was proved first in [31] for the time discrete scheme and afterwards in [5] for a fully discrete finite element (FE) scheme.

On the other hand, error estimates for projection methods were obtained (see [27], [28] for time discrete schemes and [10] for a fully discrete FE scheme). Basically, the Chorin-Temam scheme has order $O(k^{1/2})$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and $O(k)$ in $l^2(\mathbf{L}^2)$ for the velocity, and $O(k^{1/2})$ in $l^2(L^2)$ for the pressure. For the incremental-pressure scheme, these error estimates are improved in [27] and [28] to order $O(k)$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ for the velocity and $O(k)$ in $l^2(L^2)$ for the pressure (although this last estimate is proved only for the linear problem). In fact, these optimal error estimates are extended in [10] to a fully discrete FE-stable scheme (see (21) below) under the constraint $k^2 \leq Ch$ in 3D domains or $k^2 \leq \alpha(1 + \log(h^{-1}))$ in 2D ones. The argument done in [10] is based on the direct comparison between an appropriate spatial interpolation of the exact solution and the fully discrete scheme.

By the contrary, in this paper, we will obtain optimal error estimates without imposing restrictions on h and k for a FE decoupled scheme different from scheme studied in [10] (which was not decoupled because the projection step is solved by means of a mixed velocity-pressure formulation). The argument used now is also different from [10], because the corresponding time discrete scheme will be introduced as an intermediate problem. This argument has already been used in [16, 17, 18] for a different splitting scheme (with decomposition of viscosity) applied to Navier-Stokes equations.

The particular property that some projection methods (without and with incremental pressure) can be rewritten as segregated methods (decoupling velocity and pressure), was observed in [26, 27]. For a segregated fully discrete FE scheme based on the non-incremental projection method, the convergence and sub-optimal error estimates $O(k^{1/2} + h)$ for the pressure have been obtained in [3], without imposing inf-sup condition, but under the double constraint $\alpha h^2 \leq k \leq \beta h^2$.

In this paper, we obtain optimal order $O(k + h)$ for the velocity and pressure, without imposing constraints on h and k , for a time segregated scheme with first-order inf-sup stable FE spaces. Up to our knowledge, optimal first order for the pressure of a fully segregated scheme for the Navier-Stokes problem have not been proved before.

Ideas of this paper are being used to design a segregated second order in time scheme ([19]).

This paper is organized as follows:

In Section 1, we study the time discrete scheme (see Algorithm 1 below). Firstly, the stability of this scheme is deduced, and we introduce the discrete in time problems satisfied by errors and the regularity hypotheses that must be imposed on the exact solution. Afterwards, we obtain $O(k)$ accuracy for the velocity in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$. As a consequence, the velocity is bounded in

$l^\infty(\mathbf{H}^1)$. Then, we deduce $O(k)$ for the discrete in time derivative of velocities in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$. Finally, $O(k)$ for the velocity in $l^\infty(\mathbf{H}^1)$ and for the pressure in $l^\infty(L^2)$ hold.

Section 2 is devoted to study the fully discrete FE scheme (see Algorithm 2 below). We present the FE-stable spaces and their approximation properties, the fully discrete segregated scheme and the problems satisfied by the errors (comparing the time discrete Algorithm 1 with the fully discrete Algorithm 2). With respect to the spatial error estimates, firstly we obtain $O(h)$ for the velocity in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$. Then, the velocity is bounded in $l^\infty(\mathbf{H}^1)$. Afterwards, by using some additional estimates for the time discrete scheme, $O(h)$ for the discrete in time derivative of velocity in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ is obtained. Finally, $O(h)$ for the velocity in $l^\infty(\mathbf{H}^1)$ and for the pressure in $l^\infty(L^2)$ are deduced.

In Section 3, some numerical simulations are presented, showing first order accuracy in time for velocity and pressure. These simulations are also compared with the segregated versions of the rotational, consistent and penalty-projection schemes.

Finally, some conclusions are given in Section 4.

In this paper, the following discrete Gronwall's lemma will be used ([22, p. 369]):

Lemma 1 (*Discrete Gronwall inequality*) *Let k, B and a_m, b_m, c_m, γ_m be nonnegative numbers. If we assume*

$$a_{r+1} + k \sum_{m=0}^r b_m \leq k \sum_{m=0}^r \gamma_m a_m + k \sum_{m=0}^r c_m + B \quad \forall r \geq 0,$$

then, one has

$$a_{r+1} + k \sum_{m=0}^r b_m \leq \exp \left(k \sum_{m=0}^r \gamma_m \right) \left\{ k \sum_{m=0}^r c_m + B \right\} \quad \forall r \geq 0.$$

1 Time discrete scheme (Algorithm 1)

The norm and inner product in $L^2(\Omega)$ will be denoted by $|\cdot|$ and (\cdot, \cdot) , whereas the norm in $H_0^1(\Omega)$ of the gradient in $L^2(\Omega)$ will be denoted by $\|\cdot\|$. Any other norm in a space X will be denoted by $\|\cdot\|_X$

Let us to introduce the standard Hilbert spaces in the Navier-Stokes framework:

$$\begin{aligned} \mathbf{H} &= \{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n}_{\partial\Omega} = 0 \}, \\ \mathbf{V} &= \{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega \}, \end{aligned}$$

where $\mathbf{n}_{\partial\Omega}$ denotes the normal outwards vector to $\partial\Omega$.

In the sequel, the following standard skew-symmetric form of the convective term will be used:

$$C(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \cdot \nabla) \mathbf{v} + \frac{1}{2} (\nabla \cdot \mathbf{u}) \mathbf{v} \quad \forall \mathbf{u} \in \mathbf{H}_0^1, \mathbf{v} \in \mathbf{H}^1,$$

and the corresponding trilinear form

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} C(\mathbf{u}, \mathbf{v}) \cdot \mathbf{w} = \int_{\Omega} \left\{ (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} + \frac{1}{2} (\nabla \cdot \mathbf{u}) \mathbf{v} \cdot \mathbf{w} \right\}, \quad \forall \mathbf{u} \in \mathbf{H}_0^1, \mathbf{v} \in \mathbf{H}^1, \mathbf{w} \in \mathbf{H}^1$$

or equivalently

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2} \int_{\Omega} \left\{ (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} - (\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{v} \right\} = - \int_{\Omega} \left\{ (\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{v} + \frac{1}{2} (\nabla \cdot \mathbf{u}) \mathbf{v} \cdot \mathbf{w} \right\}.$$

Previous equalities hold even in the fully discrete case, hence we can use, in the sequel, any of these three possibilities.

The trilinear form $c(\cdot, \cdot, \cdot)$ satisfies

$$c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \forall \mathbf{u} \in \mathbf{H}_0^1, \quad \forall \mathbf{v} \in \mathbf{H}^1, \quad (1)$$

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C \begin{cases} \|\mathbf{u}\| \|\mathbf{v}\|_{W^{1,3} \cap L^\infty} \|\mathbf{w}\| \\ \|\mathbf{u}\|_{L^3} \|\mathbf{v}\| \|\mathbf{w}\| \end{cases}$$

where the role of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ can be interchanged, using the appropriate expression of $c(\cdot, \cdot, \cdot)$.

For simplicity and without loss of generality, we fix the viscosity constant $\nu = 1$.

1.1 Description of the time scheme (Algorithm 1)

Given $(\mathbf{f}^m = \mathbf{f}(t_m))_{m=1}^M$, we define an approximation $(\mathbf{u}^m, p^m)_{m=1}^M$ of the solution (\mathbf{u}, p) of (P) at time $t = t_m$, by means of an incremental pressure projection scheme of Van-Kan type [23], splitting the nonlinearity $(\mathbf{u} \cdot \nabla) \mathbf{u}$ and the diffusion term $-\Delta \mathbf{u}$ to the incompressibility condition $\nabla \cdot \mathbf{u} = 0$. Moreover, an explicit pressure term is introduced in the convection-diffusion problem for the velocity (Sub-step 1), with a pressure-correction in the divergence-free projection step (Sub-step 2). See Algorithm 1 for a description of the time scheme.

Notice that the convection term has been taken in $(S_1)^{m+1}$ in the semi-implicit linear form $C(\tilde{\mathbf{u}}^m, \tilde{\mathbf{u}}^{m+1})$. On the other hand, adding $(S_1)^{m+1}$ and $(S_2)^{m+1}$, we arrive at

$$(S_3)^{m+1} \quad \begin{cases} \frac{1}{k} (\mathbf{u}^{m+1} - \mathbf{u}^m) + C(\tilde{\mathbf{u}}^m, \tilde{\mathbf{u}}^{m+1}) - \Delta \tilde{\mathbf{u}}^{m+1} + \nabla p^{m+1} = \mathbf{f}^{m+1} & \text{in } \Omega, \\ \tilde{\mathbf{u}}^{m+1}|_{\partial\Omega} = 0, \quad \nabla \cdot \mathbf{u}^{m+1} = 0 & \text{in } \Omega. \end{cases}$$

In fact $(S_3)^{m+1}$ can be viewed as consistence relations, because if $\tilde{\mathbf{u}}^{m+1}$ and \mathbf{u}^{m+1} converge to the same limit velocity \mathbf{u} as k goes to zero, then taking limits in $(S_3)^{m+1}$, one has at least formally that \mathbf{u} will be a solution of the exact problem (P) .

Now, some remarks about Sub-step 2 are in order:

- Sub-step 2 can be viewed as a projection step. In fact, $\mathbf{u}^{m+1} = P_{\mathbf{H}} \tilde{\mathbf{u}}^{m+1}$ where $P_{\mathbf{H}}$ is the $L^2(\Omega)$ -projector onto \mathbf{H} , because $(S_2)^{m+1}$ implies in particular

$$(\mathbf{u}^{m+1} - \tilde{\mathbf{u}}^{m+1}, \mathbf{u}) = 0 \quad \forall \mathbf{u} \in \mathbf{H}.$$

Algorithm 1 Time discrete algorithm

Initialization: Let p^0 be given and to take $\mathbf{u}^0 = \tilde{\mathbf{u}}^0 = \mathbf{u}(0)(= \mathbf{u}_0)$.

Step of time $m + 1$: Let \mathbf{u}^m , $\tilde{\mathbf{u}}^m$ and p^m be given.

Sub-step 1: Find $\tilde{\mathbf{u}}^{m+1} : \Omega \rightarrow \mathbb{R}^3$ solving

$$(S_1)^{m+1} \quad \begin{cases} \frac{1}{k}(\tilde{\mathbf{u}}^{m+1} - \mathbf{u}^m) + C(\tilde{\mathbf{u}}^m, \tilde{\mathbf{u}}^{m+1}) - \Delta \tilde{\mathbf{u}}^{m+1} + \nabla p^m = \mathbf{f}^{m+1} & \text{in } \Omega, \\ \tilde{\mathbf{u}}^{m+1}|_{\partial\Omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

Sub-step 2: Find $\mathbf{u}^{m+1} : \Omega \rightarrow \mathbb{R}^3$ and $p^{m+1} : \Omega \rightarrow \mathbb{R}$ solution of

$$(S_2)^{m+1} \quad \begin{cases} \frac{1}{k}(\mathbf{u}^{m+1} - \tilde{\mathbf{u}}^{m+1}) + \nabla(p^{m+1} - p^m) = 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{u}^{m+1} = 0 & \text{in } \Omega, \quad \mathbf{u}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{cases}$$

- By using $\nabla \cdot \mathbf{u}^{m+1} = 0$ in Ω and $\mathbf{u}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} = 0$, one has the orthogonality property

$$(\mathbf{u}^{m+1}, \nabla q) = 0 \quad \forall q \in H^1(\Omega). \quad (2)$$

- It is well known that Sub-step 2 is equivalent to the following two (decoupled) problems:

1. Find $p^{m+1} : \Omega \rightarrow \mathbb{R}$ such that

$$(S_2)_a^{m+1} \quad \begin{cases} k \Delta(p^{m+1} - p^m) = \nabla \cdot \tilde{\mathbf{u}}^{m+1} & \text{in } \Omega \\ k \nabla(p^{m+1} - p^m) \cdot \mathbf{n}|_{\partial\Omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

2. Find $\mathbf{u}^{m+1} : \Omega \rightarrow \mathbb{R}^3$ as

$$(S_2)_b^{m+1} \quad \mathbf{u}^{m+1} = \tilde{\mathbf{u}}^{m+1} - k \nabla(p^{m+1} - p^m) \quad \text{in } \Omega.$$

1.2 Unconditional stability and convergence of Algorithm 1

Lemma 2 (Continuous dependence of the projection step) .

a) (Continuous dependence with respect to \mathbf{L}^2) If $\tilde{\mathbf{u}}^{m+1}$ and $\mathbf{u}^m \in \mathbf{L}^2(\Omega)$, then there exists an unique $\mathbf{u}^{m+1} \in \mathbf{H}$ solution of $(S_2)^{m+1}$. Moreover,

$$|\tilde{\mathbf{u}}^{m+1}|^2 = |\mathbf{u}^{m+1}|^2 + |k \nabla(p^{m+1} - p^m)|^2 \quad (3)$$

$$|\mathbf{u}^{m+1} - \tilde{\mathbf{u}}^{m+1}| \leq |\tilde{\mathbf{u}}^{m+1} - \mathbf{u}^m|. \quad (4)$$

b) (Continuous dependence with respect to \mathbf{H}^1) If $\tilde{\mathbf{u}}^{m+1} \in \mathbf{H}_0^1(\Omega)$ then $\mathbf{u}^{m+1} \in \mathbf{H}^1(\Omega) \cap \mathbf{H}$. Moreover,

$$\|\mathbf{u}^{m+1}\| \leq C \|\tilde{\mathbf{u}}^{m+1}\|.$$

Proof.

a) Since $\mathbf{u}^{m+1} = P_{\mathbf{H}}\tilde{\mathbf{u}}^{m+1}$, one has (3). Moreover, estimate (4) can be obtained directly from the best approximation property of the L^2 -projection:

$$|\mathbf{u}^{m+1} - \tilde{\mathbf{u}}^{m+1}| = \min_{\mathbf{u} \in \mathbf{H}} |\mathbf{u} - \tilde{\mathbf{u}}^{m+1}|.$$

b) By applying the $H^2(\Omega)$ -regularity of problem $(S_2)_a^{m+1}$, there exists a unique $p^{m+1} - p^m \in H^2 \cap L_0^2$ satisfying

$$k \|\nabla(p^{m+1} - p^m)\|_{H^1} \leq C \|\tilde{\mathbf{u}}^{m+1}\|.$$

Therefore, $\mathbf{u}^{m+1} \in \mathbf{H}^1(\Omega)$ and

$$\|\mathbf{u}^{m+1}\| \leq C \left\{ \|\tilde{\mathbf{u}}^{m+1}\| + k \|\nabla(p^{m+1} - p^m)\|_{H^1} \right\} \leq C \|\tilde{\mathbf{u}}^{m+1}\|.$$

This estimate can be understood as the \mathbf{H}^1 -stability of the L^2 -projector onto \mathbf{H} . ■

Lemma 3 (Stability of Algorithm 1) *Let $\mathbf{f} \in L^2(0, T; \mathbf{H}^{-1}(\Omega))$ ($\mathbf{H}^{-1}(\Omega)$ being the dual space of $\mathbf{H}_0^1(\Omega)$) and $\mathbf{u}_0 \in \mathbf{H}$. Assuming the following constraint on the initial discrete pressure $k |\nabla p^0| \leq C_0$, then there exists a constant $C = C(C_0, \mathbf{u}_0, \mathbf{f}, \Omega) > 0$ such that,*

$$\begin{aligned} |\tilde{\mathbf{u}}^{r+1}|^2 + |\mathbf{u}^{r+1}|^2 + |k \nabla p^{r+1}|^2 &\leq C, \quad \forall r = 0, \dots, M-1, \\ k \sum_{m=0}^{M-1} \{ \|\tilde{\mathbf{u}}^{m+1}\|^2 + \|\mathbf{u}^{m+1}\|^2 \} &\leq C. \end{aligned}$$

Proof. We only give here an outline of the proof, which follows the same lines given in the proof of Theorem 7 below. By making

$$2k \left((S_1)^{m+1}, \tilde{\mathbf{u}}^{m+1} \right) + k \left((S_2)^{m+1}, \tilde{\mathbf{u}}^{m+1} + \mathbf{u}^{m+1} + k (\nabla p^{m+1} + \nabla p^m) \right),$$

and using orthogonality property (2):

$$|\mathbf{u}^{m+1}|^2 + |k \nabla p^{m+1}|^2 - |\mathbf{u}^m|^2 - |k \nabla p^m|^2 + |\tilde{\mathbf{u}}^{m+1} - \mathbf{u}^m|^2 + 2k \|\tilde{\mathbf{u}}^{m+1}\|^2 = 2k (\mathbf{f}^{m+1}, \tilde{\mathbf{u}}^{m+1}),$$

hence, by using the discrete Gronwall's Lemma (Lemma 1):

$$\|\mathbf{u}^{m+1}\|_{l^\infty(\mathbf{L}^2)} + \|k \nabla p^{m+1}\|_{l^\infty(L^2)} + \|\tilde{\mathbf{u}}^{m+1}\|_{l^2(\mathbf{H}^1)} \leq C \quad \text{and} \quad \sum_{m \geq 0} |\tilde{\mathbf{u}}^{m+1} - \mathbf{u}^m|^2 \leq C.$$

Now, accounting Lemma 2, the following supplementary stability estimates hold:

$$\|\tilde{\mathbf{u}}^{m+1}\|_{l^\infty(\mathbf{L}^2)} \leq C \quad \text{and} \quad \|\mathbf{u}^{m+1}\|_{l^2(\mathbf{H}^1)} \leq C. \quad \text{■}$$

Starting from the previous stability estimates and taking limits as $k \downarrow 0$ in $(S_3)^{m+1}$, the convergence of the velocity approximations have already been established (for instance, see [32]). Concretely, defining $\mathbf{u}_k : (0, T] \rightarrow \mathbf{H} \cap \mathbf{H}^1(\Omega)$ as the piecewise constant functions taking the value \mathbf{u}^{m+1} in $(t_m, t_{m+1}]$, the following result holds:

Proposition 4 (Convergence of Algorithm 1) *Under conditions of Lemma 3, there exists a subsequence (k') of (k) , and a weak solution $\mathbf{u} \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$ of (P) in $(0, T)$, such that: $\mathbf{u}_{k'} \rightarrow \mathbf{u}$ weakly-* in $L^\infty(0, T; \mathbf{H})$, weakly in $L^2(0, T; \mathbf{H}^1(\Omega) \cap \mathbf{H})$ and strongly in $L^2(0, T; \mathbf{H})$, as $k' \downarrow 0$.*

1.3 Differential problems satisfied by the errors

We will obtain error estimates (for velocity and pressure) with respect to a sufficiently regular (and unique) solution (\mathbf{u}, p) of (P) . For this, we introduce the following notations for the errors in $t = t_{m+1}$:

$$\tilde{\mathbf{e}}^{m+1} := \mathbf{u}(t_{m+1}) - \tilde{\mathbf{u}}^{m+1}, \quad \mathbf{e}^{m+1} := \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}, \quad e_p^{m+1} := p(t_{m+1}) - p^{m+1},$$

and for the discrete in time derivative of errors

$$\delta_t \mathbf{e}^{m+1} := \frac{\mathbf{e}^{m+1} - \mathbf{e}^m}{k}, \quad \delta_t \tilde{\mathbf{e}}^{m+1} := \frac{\tilde{\mathbf{e}}^{m+1} - \tilde{\mathbf{e}}^m}{k}.$$

Subtracting $(S_1)^{m+1}$ with the momentum system of (P) at $t = t_{m+1}$, using the integral rest and manipulating the convective terms, one has:

$$(E_1)^{m+1} \quad \begin{cases} \frac{1}{k}(\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m) - \Delta \tilde{\mathbf{e}}^{m+1} + \nabla(e_p^m + k \delta_t p(t_{m+1})) = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} & \text{in } \Omega, \\ \tilde{\mathbf{e}}^{m+1}|_{\partial\Omega} = \mathbf{0}, \end{cases}$$

where

$$\mathcal{E}^{m+1} = -\frac{1}{k} \int_{t_m}^{t_{m+1}} (t - t_m) \mathbf{u}_{tt}(t) dt - \left(\int_{t_m}^{t_{m+1}} \mathbf{u}_t \cdot \nabla \right) \mathbf{u}(t_{m+1}) := \mathcal{E}_1^{m+1} + \mathcal{E}_2^{m+1}$$

is the consistency error, and

$$\mathbf{NL}^{m+1} = -C(\tilde{\mathbf{e}}^m, \mathbf{u}(t_{m+1})) - C(\tilde{\mathbf{u}}^m, \tilde{\mathbf{e}}^{m+1})$$

are terms depending of the convective terms.

On the other hand, adding and subtracting the term $\mathbf{u}(t_{m+1})$ in $(S_2)^{m+1}$,

$$(E_2)^{m+1} \quad \begin{cases} \frac{1}{k}(\mathbf{e}^{m+1} - \tilde{\mathbf{e}}^{m+1}) + \nabla(e_p^{m+1} - e_p^m - k \delta_t p(t_{m+1})) = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \mathbf{e}^{m+1} = 0 & \text{in } \Omega, \quad \mathbf{e}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Finally, adding $(E_1)^{m+1}$ and $(E_2)^{m+1}$, we arrive at:

$$(E_3)^{m+1} \quad \begin{cases} \frac{1}{k}(\mathbf{e}^{m+1} - \mathbf{e}^m) - \Delta \tilde{\mathbf{e}}^{m+1} + \nabla e_p^{m+1} = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} & \text{in } \Omega, \\ \nabla \cdot \mathbf{e}^{m+1} = 0 & \text{in } \Omega, \quad \mathbf{e}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Lemma 5 (Continuous dependence of the projection errors) *The following inequalities hold*

$$\begin{aligned} |\tilde{\mathbf{e}}^{m+1}|^2 &= |\mathbf{e}^{m+1}|^2 + |k \nabla (e_p^{m+1} - e_p^m - k \delta_t p(t_{m+1}))|^2, \\ |\mathbf{e}^{m+1} - \tilde{\mathbf{e}}^{m+1}| &\leq |\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m| \\ \|\mathbf{e}^{m+1}\| &\leq C \|\tilde{\mathbf{e}}^{m+1}\|. \end{aligned} \tag{5}$$

Proof. The proof is similar to Lemma 2, by using that $\mathbf{e}^{m+1} = P_{\mathbf{H}} \tilde{\mathbf{e}}^{m+1}$. ■

1.4 Regularity hypotheses.

We will assume the following regularity hypothesis on Ω :

(H0) $\Omega \subset \mathbb{R}^3$ such that Poisson problems in Ω have $\mathbf{H}^2(\Omega)$ -regularity.

In order to obtain the different error estimates, the following regularity hypotheses for the (unique) solution (\mathbf{u}, p) of (P) will be appearing:

$$\textbf{(H1)} \quad \mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V}), \quad p_t \in L^2(H^1), \quad \mathbf{u}_t \in L^2(\mathbf{L}^2), \quad \mathbf{u}_{tt} \in L^2(\mathbf{H}^{-1})$$

$$\textbf{(H2)} \quad p_{tt} \in L^2(H^1), \quad \mathbf{u}_t \in L^\infty(\mathbf{L}^3) \cap L^2(\mathbf{H}^1), \quad \mathbf{u}_{tt} \in L^2(\mathbf{L}^2), \quad \mathbf{u}_{ttt} \in L^2(\mathbf{H}^{-1})$$

$$\textbf{(H3)} \quad \mathbf{u}_{tt} \in L^\infty(\mathbf{H}^{-1})$$

Remark 6 *Unfortunately, to obtain hypotheses (H1)-(H3) is necessary to assume that $\mathbf{u}_t(0) \in \mathbf{H}^1$, which implies a non-local compatibility condition for the data \mathbf{u}_0 and $\mathbf{f}(0)$. In particular, it is proved in [21] that (H1)-(H3) is satisfied (at least locally in time), if there exists $p_0 \in H^1(\Omega)$ (the initial pressure) solution of the following overdetermined Neumann problem*

$$\begin{cases} \Delta p_0 = \nabla \cdot (\mathbf{f}(0) - (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0) & \text{in } \Omega, \\ \nabla p_0|_{\partial\Omega} = (\Delta \mathbf{u}_0 + \mathbf{f}(0) - (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0)|_{\partial\Omega}. \end{cases}$$

which in practice is hard to fulfill (see [21]).

In [24], error estimates for the (non-incremental) Chorin-Temam projection scheme are deduced without requiring this non-local compatibility condition, arriving at the optimal order $O(k)$ in $l^\infty(\mathbf{L}^2)$ for the velocity and in $l^\infty(H^{-1})$ for the pressure, where a weight at the initial time steps must be included to deduce the optimal order for the pressure (only possible in a negative norm).

Nevertheless, for the incremental scheme Algorithm 1 it is not clear how to avoid this compatibility on the data using adequate weights at the initial time steps.

1.5 $O(k)$ -error estimates for both velocities

Theorem 7 *Under conditions of Lemma 3, (H1) and the bound for the initial error pressure $|\nabla e_p^0| \leq C$, the following error estimates hold:*

$$\|\tilde{\mathbf{e}}^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} \leq C k, \quad \|e_p^{m+1}\|_{l^\infty(H^1)} \leq C, \quad (6)$$

$$\|\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m\|_{l^2(\mathbf{L}^2)} + \|\mathbf{e}^{m+1} - \tilde{\mathbf{e}}^{m+1}\|_{l^2(\mathbf{L}^2)} \leq C k^{3/2}. \quad (7)$$

Proof. The proof follows similar lines of [10] and [27].

By multiplying $(E_1)^{m+1}$ by $2k\tilde{\mathbf{e}}^{m+1}$ and integrating in Ω , one has:

$$\begin{aligned} & |\tilde{\mathbf{e}}^{m+1}|^2 - |\mathbf{e}^m|^2 + |\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + 2k\|\tilde{\mathbf{e}}^{m+1}\|^2 + 2k\left(\nabla e_p^m, \tilde{\mathbf{e}}^{m+1}\right) \\ &= 2k\left(\mathcal{E}^{m+1} + \mathbf{NL}^{m+1}, \tilde{\mathbf{e}}^{m+1}\right) - 2k^2\left(\nabla \delta_t p(t_{m+1}), \tilde{\mathbf{e}}^{m+1}\right). \end{aligned} \quad (8)$$

On the other hand, multiplying $(E_2)^{m+1}$ by $k(\mathbf{e}^{m+1} + \tilde{\mathbf{e}}^{m+1}) + k^2(\nabla e_p^{m+1} + \nabla e_p^m)$ and using that $(\mathbf{e}^{m+1}, \nabla e_p^{m+1}) = 0 = (\mathbf{e}^{m+1}, \nabla e_p^m) = (\mathbf{e}^{m+1}, \nabla \delta_t p(t_{m+1}))$ (see (2)), we obtain

$$\begin{aligned} & |\mathbf{e}^{m+1}|^2 - |\tilde{\mathbf{e}}^{m+1}|^2 + |k\nabla e_p^{m+1}|^2 - |k\nabla e_p^m|^2 - 2k\left(\tilde{\mathbf{e}}^{m+1}, \nabla e_p^m\right) \\ &= k^2\left(\tilde{\mathbf{e}}^{m+1}, \nabla \delta_t p(t_{m+1})\right) + k^3\left(\nabla \delta_t p(t_{m+1}), \nabla e_p^{m+1} + \nabla e_p^m\right). \end{aligned} \quad (9)$$

By adding (8) and (9), the term $2k(\tilde{\mathbf{e}}^{m+1}, \nabla e_p^m)$ vanish, obtaining

$$\begin{aligned} & |\mathbf{e}^{m+1}|^2 - |\mathbf{e}^m|^2 + |k\nabla e_p^{m+1}|^2 - |k\nabla e_p^m|^2 + |\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + 2k\|\tilde{\mathbf{e}}^{m+1}\|^2 \\ & \leq 2k\left(\mathcal{E}^{m+1} + \mathbf{NL}^{m+1}, \tilde{\mathbf{e}}^{m+1}\right) - k^2\left(\nabla \delta_t p(t_{m+1}), \tilde{\mathbf{e}}^{m+1}\right) \\ & \quad + k^3\left(\nabla \delta_t p(t_{m+1}), \nabla e_p^{m+1} + \nabla e_p^m\right). \end{aligned} \quad (10)$$

The consistency error can be bounded as follows:

$$2k\left(\mathcal{E}_1^{m+1}, \tilde{\mathbf{e}}^{m+1}\right) \leq \frac{k}{3}\|\tilde{\mathbf{e}}^{m+1}\|^2 + Ck^2 \int_{t_m}^{t_{m+1}} \|\mathbf{u}_{tt}\|_{\mathbf{H}^{-1}}^2 dt,$$

$$2k\left(\mathcal{E}_2^{m+1}, \tilde{\mathbf{e}}^{m+1}\right) \leq 2k\left(\int_{t_m}^{t_{m+1}} |\mathbf{u}_t| \right) \|\nabla \mathbf{u}(t_{m+1})\|_{L^3} \|\tilde{\mathbf{e}}^{m+1}\|_{L^6} \leq \frac{k}{3}\|\tilde{\mathbf{e}}^{m+1}\|^2 + Ck^2 \int_{t_m}^{t_{m+1}} |\mathbf{u}_t|^2.$$

By using the antisymmetry property $c(\tilde{\mathbf{u}}^m, \tilde{\mathbf{e}}^{m+1}, \tilde{\mathbf{e}}^{m+1}) = 0$ and equality (5), we bound the convective terms as follows:

$$\begin{aligned} 2k\left(\mathbf{NL}^{m+1}, \tilde{\mathbf{e}}^{m+1}\right) &= 2kc\left(\tilde{\mathbf{e}}^m, \mathbf{u}(t_{m+1}), \tilde{\mathbf{e}}^{m+1}\right) \leq \frac{k}{3}\|\tilde{\mathbf{e}}^{m+1}\|^2 + Ck\|\mathbf{u}(t_{m+1})\|_{L^\infty \cap W^{1,3}}^2 |\tilde{\mathbf{e}}^m|^2 \\ &\leq \frac{k}{3}\|\tilde{\mathbf{e}}^{m+1}\|^2 + Ck|\mathbf{e}^m|^2 + Ck\left(|k\nabla e_p^m|^2 + |k\nabla e_p^{m-1}|^2 + k^2|\nabla \delta_t p(t_m)|^2\right) \end{aligned}$$

Now, by using that $(\mathbf{e}^m, \nabla \delta_t p(t_{m+1})) = 0$, we bound the third term at RHS of (10):

$$-k^2\left(\tilde{\mathbf{e}}^{m+1}, \nabla \delta_t p(t_{m+1})\right) = -k^2\left(\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m, \nabla \delta_t p(t_{m+1})\right) \leq \frac{1}{4}|\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + Ck^4|\nabla \delta_t p(t_{m+1})|^2.$$

Finally, we bound the last term at RHS of (10):

$$k^3 \left(\nabla \delta_t p(t_{m+1}), \nabla e_p^{m+1} + \nabla e_p^m \right) = k^3 \left(\nabla \delta_t p(t_{m+1}), \nabla e_p^{m+1} - \nabla e_p^m \right) + k^3 \left(\nabla \delta_t p(t_{m+1}), 2 \nabla e_p^m \right) = I_1 + I_2$$

By using $(E_2)^{m+1}$, the I_1 -term can be rewritten as

$$\begin{aligned} I_1 &= k^2 \left(\nabla \delta_t p(t_{m+1}), \tilde{\mathbf{e}}^{m+1} - \mathbf{e}^{m+1} \right) + k^4 |\nabla \delta_t p(t_{m+1})|^2 \\ &= k^2 \left(\nabla \delta_t p(t_{m+1}), \tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m \right) + k^4 |\nabla \delta_t p(t_{m+1})|^2 \\ &\leq \frac{1}{4} |\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + C k^3 \int_{t_m}^{t_{m+1}} |\nabla p_t|^2 \end{aligned}$$

We bound I_2 as:

$$I_2 \leq C k |k \nabla e_p^m|^2 + C k^3 |\nabla \delta_t p(t_{m+1})|^2 \leq C k |k \nabla e_p^m|^2 + C k^2 \int_{t_m}^{t_{m+1}} |\nabla p_t|^2$$

By applying these bounds in (10),

$$\begin{aligned} &|\mathbf{e}^{m+1}|^2 - |\mathbf{e}^m|^2 + |k \nabla e_p^{m+1}|^2 - |k \nabla e_p^m|^2 + \frac{1}{2} |\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + k \|\tilde{\mathbf{e}}^{m+1}\|^2 \leq C k |\mathbf{e}^m|^2 \\ &+ C k^2 \int_{t_m}^{t_{m+1}} \left(\|\mathbf{u}_{tt}\|_{\mathbf{H}^{-1}}^2 + |\mathbf{u}_t|^2 + |\nabla p_t|^2 \right) dt + C k \left(|k \nabla e_p^m|^2 + |k \nabla e_p^{m-1}|^2 \right). \end{aligned}$$

Adding up from $m = 1$ to r , and applying the discrete Gronwall inequality, we arrive at:

$$\|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)} + \|\tilde{\mathbf{e}}^{m+1}\|_{l^2(\mathbf{H}^1)} \leq C k, \quad \|\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m\|_{l^2(\mathbf{L}^2)} \leq C k^{3/2} \quad \text{and} \quad \|e_p^{m+1}\|_{l^\infty(H^1)} \leq C.$$

Finally, by applying Lemma 5, estimates $\|\tilde{\mathbf{e}}^{m+1}\|_{l^\infty(\mathbf{L}^2)} \leq C k$ and $\|\mathbf{e}^{m+1}\|_{l^2(\mathbf{H}^1)} \leq C k$ hold. \blacksquare

Notice that the error estimate $\|\tilde{\mathbf{e}}^m\|_{l^2(\mathbf{H}^1)} \leq C k$ implies in particular the uniform estimates

$$\|\tilde{\mathbf{e}}^m\|_{\mathbf{H}^1} \leq C \quad \text{and} \quad \|\tilde{\mathbf{u}}^m\|_{\mathbf{H}^1} \leq C \quad \forall m.$$

1.6 $O(k)$ -error estimates for the pressure

First, we are going to obtain error estimates for the discrete time derivative of velocity, and then the optimal order $O(k)$ for the pressure.

Lemma 8 (Continuous dependence of discrete derivatives for the projection step) *It holds*

$$\begin{aligned} |\delta_t \tilde{\mathbf{e}}^{m+1}|^2 &= |\delta_t \mathbf{e}^{m+1}|^2 + |k \nabla \delta_t (e_p^{m+1} - e_p^m - k \delta_t p(t_{m+1}))|^2, \\ |\delta_t \mathbf{e}^{m+1} - \delta_t \tilde{\mathbf{e}}^{m+1}| &\leq |\delta_t \tilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m|, \\ \|\delta_t \mathbf{e}^{m+1}\| &\leq C \|\delta_t \tilde{\mathbf{e}}^{m+1}\|. \end{aligned} \tag{11}$$

Proof. The proof is similar to Lemma 2 and Lemma 5, using that $\delta_t \mathbf{e}^{m+1} = P_{\mathbf{H}}(\delta_t \tilde{\mathbf{e}}^{m+1})$. \blacksquare

Theorem 9 Assuming hypotheses of Theorem 7, **(H2)** and the following constraints on the first-step approximation

$$|\delta_t \mathbf{e}^1| + |k \nabla \delta_t e_p^1| \leq C k \quad \text{and} \quad \|\delta_t \tilde{\mathbf{e}}^1\| \leq C \sqrt{k},$$

one has

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} + \|\delta_t \tilde{\mathbf{e}}^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} \leq C k \quad \text{and} \quad \|\delta_t e_p^{m+1}\|_{l^\infty(H^1)} \leq C.$$

Proof. By making $\delta_t(E_1)^{m+1}$ and $\delta_t(E_2)^{m+1}$:

$$(D_1)^{m+1} \quad \frac{\delta_t \tilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m}{k} - \Delta \delta_t \tilde{\mathbf{e}}^{m+1} + \nabla(\delta_t e_p^m + k \delta_t \delta_t p(t_{m+1})) = \delta_t \mathcal{E}^{m+1} + \delta_t \mathbf{NL}^{m+1}$$

where $\delta_t \delta_t p(t_{m+1}) = \frac{1}{k}(\delta_t p(t_{m+1}) - \delta_t p(t_m))$, and

$$(D_2)^{m+1} \quad \frac{\delta_t \mathbf{e}^{m+1} - \delta_t \tilde{\mathbf{e}}^{m+1}}{k} + \nabla(\delta_t e_p^{m+1} - \delta_t e_p^m - k \delta_t \delta_t p(t_{m+1})) = 0.$$

The proof follows similar lines of Theorem 7. Multiplying $(D_1)^{m+1}$ by $2k \delta_t \tilde{\mathbf{e}}^{m+1}$, we get:

$$\begin{aligned} & |\delta_t \tilde{\mathbf{e}}^{m+1}|^2 - |\delta_t \mathbf{e}^m|^2 + |\delta_t \tilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m|^2 + 2k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + 2k \left(\nabla \delta_t e_p^m, \delta_t \tilde{\mathbf{e}}^{m+1} \right) \\ &= 2k \left(\delta_t \mathcal{E}^{m+1} + \delta_t \mathbf{NL}^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) - 2k^2 \left(\nabla \delta_t \delta_t p(t_{m+1}), \delta_t \tilde{\mathbf{e}}^{m+1} \right). \end{aligned} \quad (12)$$

On the other hand, multiplying $(D_2)^{m+1}$ by $k(\delta_t \mathbf{e}^{m+1} + \delta_t \tilde{\mathbf{e}}^{m+1}) + k^2(\nabla \delta_t e_p^{m+1} + \nabla \delta_t e_p^m)$,

$$\begin{aligned} & |\delta_t \mathbf{e}^{m+1}|^2 - |\delta_t \tilde{\mathbf{e}}^{m+1}|^2 + |k \nabla \delta_t e_p^{m+1}|^2 - |k \nabla \delta_t e_p^m|^2 - 2k \left(\delta_t \tilde{\mathbf{e}}^{m+1}, \nabla \delta_t e_p^m \right) \\ &= k^2 \left(\nabla \delta_t \delta_t p(t_{m+1}), \delta_t \tilde{\mathbf{e}}^{m+1} \right) + k^3 \left(\nabla \delta_t \delta_t p(t_{m+1}), \nabla \delta_t e_p^{m+1} + \nabla \delta_t e_p^m \right). \end{aligned} \quad (13)$$

By adding (12) and (13), the term $2k \left(\delta_t \tilde{\mathbf{e}}^{m+1}, \nabla \delta_t e_p^m \right)$ cancels, arriving at

$$\begin{aligned} & |\delta_t \mathbf{e}^{m+1}|^2 - |\delta_t \mathbf{e}^m|^2 + |\delta_t \tilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m|^2 + |k \nabla \delta_t e_p^{m+1}|^2 - |k \nabla \delta_t e_p^m|^2 + 2k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 \\ &= 2k \left(\delta_t \mathcal{E}^{m+1} + \delta_t \mathbf{NL}^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) \\ &\quad - k^2 \left(\nabla \delta_t \delta_t p(t_{m+1}), \delta_t \tilde{\mathbf{e}}^{m+1} \right) + k^3 \left(\nabla \delta_t e_p^{m+1} + \nabla \delta_t e_p^m, \nabla \delta_t \delta_t p(t_{m+1}) \right). \end{aligned} \quad (14)$$

We bound the RHS of (14) as follows:

$$\begin{aligned} & 2k \left(\delta_t \mathcal{E}_1^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + C k^2 \int_{t_{m-1}}^{t_{m+1}} \|\mathbf{u}_{ttt}\|_{H^{-1}}^2 \\ & 2k \left(\delta_t \mathcal{E}_2^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) = 2k \left(\delta_t \mathbf{u}(t_{m+1}) \cdot \nabla (\mathbf{u}(t_{m+1}) - \mathbf{u}(t_m)), \delta_t \tilde{\mathbf{e}}^{m+1} \right) \\ & \quad + 2k \left((\delta_t \mathbf{u}(t_{m+1}) - \delta_t \mathbf{u}(t_m)) \cdot \nabla \mathbf{u}(t_m), \delta_t \tilde{\mathbf{e}}^{m+1} \right) := I_1 + I_2 \\ & I_1 \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + C k \|\delta_t \mathbf{u}(t_{m+1})\|_{L^3}^2 \left\| \int_{t_m}^{t_{m+1}} \partial_t \mathbf{u} \|_{H^1}^2 \right\| \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + C k^2 \|\mathbf{u}_t\|_{L^\infty(L^3)} \int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|_{H^1}^2 \end{aligned}$$

$$I_2 \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + C k \|\nabla \mathbf{u}(t_m)\|_{L^3}^2 \left| \delta_t \left(\int_{t_m}^{t_{m+1}} \mathbf{u}_t \right) \right|^2 \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + C k^2 \int_{t_{m-1}}^{t_{m+1}} |\mathbf{u}_{tt}|^2$$

(in the above inequality we have used estimates obtained in [28]).

Now, we bound the non-linear terms:

$$\begin{aligned} 2k \left(\delta_t \mathbf{NL}^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) &= 2k c \left(\delta_t \tilde{\mathbf{e}}^m, \mathbf{u}(t_{m+1}), \delta_t \tilde{\mathbf{e}}^{m+1} \right) + 2k c \left(\delta_t \tilde{\mathbf{u}}^m, \tilde{\mathbf{e}}^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) \\ &+ 2k c \left(\tilde{\mathbf{e}}^{m-1}, \delta_t \mathbf{u}(t_{m+1}), \delta_t \tilde{\mathbf{e}}^{m+1} \right) + 2k c \left(\tilde{\mathbf{u}}^{m-1}, \delta_t \tilde{\mathbf{e}}^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) := \sum_{i=1}^4 L_i \end{aligned}$$

$$\begin{aligned} L_1 &\leq k |\delta_t \tilde{\mathbf{e}}^m| \|\mathbf{u}(t_{m+1})\|_{L^\infty \cap W^{1,3}} \|\delta_t \tilde{\mathbf{e}}^{m+1}\| \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + C k |\delta_t \tilde{\mathbf{e}}^m|^2 \\ &\leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + C k |\delta_t \tilde{\mathbf{e}}^m|^2 + C k \left(|k \nabla \delta_t e_p^m|^2 + |k \nabla \delta_t e_p^{m-1}|^2 + C k^2 |\nabla \delta_t \delta_t p(t_m)|^2 \right) \end{aligned}$$

(here (11) is used),

$$L_2 = 2k c \left(\delta_t \tilde{\mathbf{e}}^m, \tilde{\mathbf{e}}^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) + 2k c \left(\delta_t \mathbf{u}(t_m), \tilde{\mathbf{e}}^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) := L_{21} + L_{22}$$

$$L_{21} \leq 2k \|\tilde{\mathbf{e}}^{m+1}\| \|\delta_t \tilde{\mathbf{e}}^m\|_{L^3} \|\delta_t \tilde{\mathbf{e}}^{m+1}\| \leq \varepsilon k \left(\|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + \|\delta_t \tilde{\mathbf{e}}^m\|^2 \right) + C k |\delta_t \tilde{\mathbf{e}}^m|^2$$

where we have used that $\|\tilde{\mathbf{e}}^{m+1}\| \leq C$,

$$L_{22} \leq k \|\delta_t \mathbf{u}(t_m)\|_{L^3} \|\delta_t \tilde{\mathbf{e}}^{m+1}\| \|\tilde{\mathbf{e}}^{m+1}\| \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + C k \|\tilde{\mathbf{e}}^{m+1}\|^2$$

(in the above estimate we have used the regularity $\mathbf{u}_t \in L^\infty(\mathbf{L}^3)$), and from a similar way,

$$L_3 \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 + C k \|\tilde{\mathbf{e}}^{m-1}\|^2.$$

Finally,

$$L_4 = 0.$$

Reasoning as in Theorem 7, taking into account the above estimates and choosing ε small enough, we arrive at

$$\begin{aligned} &|\delta_t \tilde{\mathbf{e}}^{m+1}|^2 - |\delta_t \tilde{\mathbf{e}}^m|^2 + \frac{1}{2} |\delta_t \tilde{\mathbf{e}}^{m+1} - \delta_t \tilde{\mathbf{e}}^m|^2 + |k \nabla \delta_t e_p^{m+1}|^2 - |k \nabla \delta_t e_p^m|^2 + k \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 \\ &\leq C k |\delta_t \tilde{\mathbf{e}}^m|^2 + C k^2 \int_{t_{m-1}}^{t_{m+1}} \left(\|\mathbf{u}_{ttt}\|_{H^{-1}}^2 + |\mathbf{u}_{tt}|^2 \right) + C k^2 \int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|^2 + \frac{k}{2} \|\delta_t \tilde{\mathbf{e}}^m\|^2 \\ &+ C k \left(\|\tilde{\mathbf{e}}^{m-1}\|^2 + \|\tilde{\mathbf{e}}^{m+1}\|^2 \right) + C k \left(|k \nabla \delta_t e_p^m|^2 + |k \nabla \delta_t e_p^{m-1}|^2 \right) + C k^2 \int_{t_{m-1}}^{t_{m+1}} |\nabla p_{tt}|^2. \end{aligned}$$

Now, by adding from $m = 1$ to r and using error estimates of Theorem 7, we arrive at

$$\begin{aligned} &|\delta_t \tilde{\mathbf{e}}^{r+1}|^2 + |k \nabla \delta_t e_p^{r+1}|^2 + \frac{1}{2} \sum_{m=1}^r |\delta_t \tilde{\mathbf{e}}^{m+1} - \delta_t \tilde{\mathbf{e}}^m|^2 + \frac{k}{2} \sum_{m=1}^r \|\delta_t \tilde{\mathbf{e}}^{m+1}\|^2 \\ &\leq |\delta_t \tilde{\mathbf{e}}^1|^2 + |k \nabla \delta_t e_p^1|^2 + \frac{k}{2} \|\delta_t \tilde{\mathbf{e}}^1\|^2 + C k \sum_{m=1}^r \left(|\delta_t \tilde{\mathbf{e}}^m|^2 + |k \nabla \delta_t e_p^m|^2 + |k \nabla \delta_t e_p^{m-1}|^2 \right) + C k^2. \end{aligned}$$

Then, applying the discrete Gronwall Lemma, we obtain the estimates

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(L^2)} \leq C k, \quad \sum_{m=1}^r |\delta_t \tilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m|^2 \leq C k^2, \quad \|\delta_t \tilde{\mathbf{e}}^{m+1}\|_{l^2(H^1)} \leq C k$$

$$\text{and} \quad \|\delta_t e_p^{m+1}\|_{l^\infty(H^1)} \leq C.$$

After that, taking into account Lemma 8,

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^2(H^1)} \leq C k \quad \sum |\delta_t \tilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^{m+1}|^2 \leq C k^2 \quad \text{and} \quad \|\delta_t \tilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)} \leq C k$$

hence the proof is finished. \blacksquare

Theorem 10 *Under hypothesis of Theorem 9 and (H3), the following error estimates hold*

$$\|\tilde{\mathbf{e}}^{m+1}\|_{l^\infty(H^1)} + \|e_p^{m+1}\|_{l^\infty(L^2)} \leq C k. \quad (15)$$

Proof.

Step 1. To prove

$$\|e_p^{m+1}\|_{l^2(L^2)} \leq C k. \quad (16)$$

We are going to deduce the estimate (16) from Theorem 9 and the continuous inf-sup condition applied to $(E_3)^{m+1}$. Indeed, rewritten $(E_3)^{m+1}$ as

$$-\nabla e_p^{m+1} = \delta_t \mathbf{e}^{m+1} - \Delta \tilde{\mathbf{e}}^{m+1} - \mathcal{E}^{m+1} - \mathbf{NL}^{m+1}, \quad e_p^{m+1} \in L_0^2(\Omega),$$

then, applying the continuous inf-sup condition

$$\begin{aligned} \|e_p^{m+1}\|_{L^2} &\leq C \left\{ \|\delta_t \mathbf{e}^{m+1}\|_{H^{-1}} + \|\tilde{\mathbf{e}}^{m+1}\| + \|\mathcal{E}^{m+1}\|_{H^{-1}} + \|\mathbf{NL}^{m+1}\|_{H^{-1}} \right\} \\ &\leq C \left\{ \|\delta_t \mathbf{e}^{m+1}\|_{H^{-1}} + \|\tilde{\mathbf{e}}^{m+1}\| + \|\tilde{\mathbf{e}}^m\| + k \|\mathbf{u}_{tt}\|_{L^\infty(0,T;H^{-1})} + k \|\mathbf{u}_t\|_{L^\infty(0,T;L^3)} \right\}, \end{aligned} \quad (17)$$

where we have used the estimate

$$\|\mathbf{NL}^{m+1}\|_{H^{-1}} \leq C(\|\tilde{\mathbf{e}}^m\| \|\mathbf{u}(t_{m+1})\|_{L^3} + \|\tilde{\mathbf{e}}^{m+1}\| \|\tilde{\mathbf{u}}^m\|_{L^3}) \leq C(\|\tilde{\mathbf{e}}^m\| + \|\tilde{\mathbf{e}}^{m+1}\|).$$

By taking into account that $\|\tilde{\mathbf{e}}^{m+1}\|_{l^2(H^1)} \leq C k$ and $\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(L^2)} \leq C k$ and hypothesis (H2) and (H3), we arrive at (16).

Step 2. To prove (15) for $\tilde{\mathbf{e}}^{m+1}$.

From $(E_3)^{m+1}$ we have

$$-\Delta \tilde{\mathbf{e}}^{m+1} = -\delta_t \mathbf{e}^{m+1} - \nabla e_p^{m+1} + \mathcal{E}^{m+1} + \mathbf{NL}^{m+1}, \quad \tilde{\mathbf{e}}^{m+1}|_{\partial\Omega} = 0.$$

Multiplying by $2k \delta_t \tilde{\mathbf{e}}^{m+1}$, we obtain

$$\begin{aligned} &|\nabla \tilde{\mathbf{e}}^{m+1}|^2 - |\nabla \tilde{\mathbf{e}}^m|^2 + |\nabla \tilde{\mathbf{e}}^{m+1} - \nabla \tilde{\mathbf{e}}^m|^2 \\ &= 2k \left(-\nabla e_p^{m+1} - \delta_t \mathbf{e}^{m+1} + \mathcal{E}^{m+1} + \mathbf{NL}^{m+1}, \delta_t \tilde{\mathbf{e}}^{m+1} \right) \\ &\leq C k |e_p^{m+1}|^2 + C k |\nabla \delta_t \tilde{\mathbf{e}}^{m+1}|^2 + C k |\delta_t \mathbf{e}^{m+1}|^2 + C k \|\mathcal{E}^{m+1}\|_{H^{-1}}^2 + C k \|\mathbf{NL}^{m+1}\|_{H^{-1}}^2 \\ &\leq C k |e_p^{m+1}|^2 + C k |\nabla \delta_t \tilde{\mathbf{e}}^{m+1}|^2 + C k |\delta_t \mathbf{e}^{m+1}|^2 + C k^3 + C k (\|\tilde{\mathbf{e}}^m\|^2 + \|\tilde{\mathbf{e}}^{m+1}\|^2) \end{aligned} \quad (18)$$

where we have bounded the two last terms at RHS of (18) as in (17). Adding (18) from $m = 0$ to r and applying the estimates of Theorems 7 and 9 and (16), we arrive at (15) for $\tilde{\mathbf{e}}^{m+1}$.

Step 3. To prove (15) for e_p^{m+1} .

By using the inequality (17) and taking into account that

$$\|\delta_t \mathbf{e}^{m+1}\|_{H^{-1}} \leq C |\delta_t \mathbf{e}^{m+1}| \leq C k \quad \text{and} \quad \|\tilde{\mathbf{e}}^{m+1}\| \leq C k,$$

we arrive at (15) for e_p^{m+1} . ■

1.7 Additional estimates

Now, we are going to obtain some H^2 stability estimates which will be necessary in next Section to get optimal error estimates in space.

Lemma 11 *Under hypotheses of Theorem 7 and (H0), one has*

$$\|\tilde{\mathbf{e}}^{m+1}\|_{\mathbf{H}^2} \leq C, \quad \forall m.$$

Proof. From the H^2 -regularity of the Poisson problem $(E_1)^{m+1}$, one has

$$\|\tilde{\mathbf{e}}^{m+1}\|_{\mathbf{H}^2}^2 \leq C \left(\left| \frac{\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m}{k} \right|^2 + |\nabla e_p^m|^2 + k^2 |\nabla \delta_t p(t_{m+1})|^2 + |\mathcal{E}^{m+1}|^2 + |\mathbf{NL}^{m+1}|^2 \right). \quad (19)$$

The first and second term of the RHS of (19) are bounded using that $|\tilde{\mathbf{e}}^{m+1} - \mathbf{e}^m| \leq C k$ from (7) and $\|e_p^{m+1}\|_{l^\infty(H^1)} \leq C$ from (6). It is easy to bound the third and the forth term of the RHS of (19). Finally, we bound the nonlinear term as follows

$$\begin{aligned} |\mathbf{NL}^{m+1}|^2 &\leq C \left(\|\tilde{\mathbf{e}}^m\|_{\mathbf{L}^\infty \cap \mathbf{W}^{1,3}}^2 \|\mathbf{u}(t_{m+1})\|^2 + \|\tilde{\mathbf{u}}^m\|^2 \|\tilde{\mathbf{e}}^{m+1}\|_{\mathbf{L}^\infty \cap \mathbf{W}^{1,3}}^2 \right) \\ &\leq C \left(\|\tilde{\mathbf{e}}^m\| \|\tilde{\mathbf{e}}^m\|_{\mathbf{H}^2} + \|\tilde{\mathbf{e}}^{m+1}\| \|\tilde{\mathbf{e}}^{m+1}\|_{\mathbf{H}^2} \right) \leq \varepsilon \left(\|\tilde{\mathbf{e}}^m\|_{\mathbf{H}^2}^2 + \|\tilde{\mathbf{e}}^{m+1}\|_{\mathbf{H}^2}^2 + C \right). \end{aligned}$$

Then, by applying these estimates in (19) and taking a small enough ε , there exists $\alpha < 1$ such that

$$\|\tilde{\mathbf{e}}^{m+1}\|_{\mathbf{H}^2}^2 \leq \alpha \|\tilde{\mathbf{e}}^m\|_{\mathbf{H}^2}^2 + C,$$

hence, by an induction process,

$$\|\tilde{\mathbf{e}}^{m+1}\|_{\mathbf{H}^2}^2 \leq \alpha^{m+1} \|\tilde{\mathbf{e}}^0\|_{\mathbf{H}^2}^2 + C (\alpha^m + \dots + \alpha + 1) \leq C$$

and the proof is concluded. ■

Remark 12 *As a consequence of the l^∞ in time estimates $\|\tilde{\mathbf{e}}^{m+1}\|_{\mathbf{H}^2} \leq C$ and $\|e_p^{m+1}\| \leq C$, $\forall m$, one also has*

$$\|\tilde{\mathbf{u}}^{m+1}\|_{\mathbf{H}^2} \leq C \quad \text{and} \quad \|p^{m+1}\| \leq C \quad \forall m.$$

On the other hand, as a direct consequence of Theorem 9, one has

$$\|\delta_t \delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)} + \|\delta_t \delta_t \tilde{\mathbf{e}}^{m+1}\|_{l^\infty(\mathbf{L}^2)} \leq C.$$

In particular, using that $\mathbf{u}_{tt} \in L^2(\mathbf{L}^2)$ (see **(H2)**), this estimate can be extended to the scheme as

$$\|\delta_t \delta_t \tilde{\mathbf{u}}^{m+1}\|_{l^2(\mathbf{L}^2)} \leq C. \quad (20)$$

Lemma 13 *Under hypotheses of Theorem 9 and **(H0)**, one has*

$$\|\delta_t \tilde{\mathbf{e}}^{m+1}\|_{l^2(\mathbf{H}^2)} \leq C.$$

In particular $\|\delta_t \tilde{\mathbf{u}}^{m+1}\|_{l^2(\mathbf{H}^2)} \leq C$.

Proof. The idea is to argue as in Lemma 11, using the H^2 -regularity of the Poisson problem $(D_1)^{m+1}$ and applying Theorem 9. ■

2 Fully discrete scheme (Algorithm 2)

In this section, we will denote by C different constants, always independent of k and h .

2.1 Finite element approximation and fully discrete scheme

We consider a segregated FE approximation of the time discrete Algorithm 1. We restrict ourselves to the case where Ω is a 2D polygon or a 3D polyhedron satisfying the regularity hypothesis **(H0)**. We consider two FE spaces $\mathbf{Y}_h \subset \mathbf{H}_0^1(\Omega)$ and $Q_h \subset H^1(\Omega) \cap L_0^2(\Omega)$ associated to a regular family of triangulations \mathcal{T}_h of the domain Ω of mesh size h (regular in the Ciarlet's sense [6]). For simplicity, we restrict \mathbf{Y}_h and Q_h to globally continuous functions and locally polynomials of degree at least 1. Finally, we will assume:

1. The inverse inequality $\|\mathbf{u}_h\| \leq C h^{-1} |\mathbf{u}_h|$ for each $\mathbf{u}_h \in \mathbf{Y}_h$ holds.
2. The stable “inf-sup” condition ([9]) for (\mathbf{Y}_h, Q_h) : There exists $\beta > 0$ independent of h such that,

$$\inf_{q_h \in Q_h \setminus \{0\}} \left(\sup_{\mathbf{v}_h \in \mathbf{Y}_h \setminus \{0\}} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\| |q_h|} \right) \geq \beta. \quad (21)$$

3. There exists some interpolation operators with the following properties:

(a) $I_h : \mathbf{L}^2 \rightarrow \mathbf{Y}_h$ such as

$$(\mathbf{u} - I_h \mathbf{u}, \nabla q_h) = 0, \quad \forall q_h \in Q_h \quad (22)$$

satisfying the approximation properties:

$$\begin{aligned} \|\mathbf{u} - I_h \mathbf{u}\|_{H^{-1}} &\leq C h \|\mathbf{u}\|_{L^2} \quad \forall \mathbf{u} \in \mathbf{L}^2(\Omega), \\ \|\mathbf{u} - I_h \mathbf{u}\|_{L^2} &\leq C h \|\mathbf{u}\|_{H^1} \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega), \\ \|\mathbf{u} - I_h \mathbf{u}\|_{H^1} &\leq C h \|\mathbf{u}\|_{H^2} \quad \forall \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \end{aligned} \quad (23)$$

and the stability property:

$$\|I_h \mathbf{u}\|_{H^1} \leq C \|\mathbf{u}\|_{H^1} \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega).$$

(b) $J_h : H^1(\Omega) \cap L_0^2(\Omega) \rightarrow Q_h$ defined by

$$\left(\nabla(J_h p - p), \nabla q_h \right) = 0 \quad \forall q_h \in Q_h,$$

satisfied the approximation property

$$\|p - J_h p\|_{L^2} \leq C h \|p\|_{H^1} \quad \forall p \in H^1(\Omega) \cap L_0^2(\Omega).$$

Remark 14 (Choice of I_h) For instance, if we consider the \mathbb{P}_1 -bubble $\times \mathbb{P}_1$ approximation to construct the space $\mathbf{Y}_h \times Q_h$, then a possible manner to choose I_h is as follows: Let \tilde{I}_h be a regularization interpolation operator (of Clément or Scott-Zhang type) onto the globally continuous and locally \mathbb{P}_1 FE space, that is $\tilde{I}_h \mathbf{u} \in C^0(\bar{\Omega})$ and $\tilde{I}_h \mathbf{u}|_T \in \mathbb{P}_1$ for each $T \in \mathcal{T}_h$. Then, \tilde{I}_h satisfies

$$\|\tilde{I}_h \mathbf{u} - \mathbf{u}\| \leq C h \|\mathbf{u}\|, \quad \|\tilde{I}_h \mathbf{u} - \mathbf{u}\| \leq C h \|\mathbf{u}\|_{H^2}. \quad (24)$$

We define $I_h \mathbf{u} = \tilde{I}_h \mathbf{u} + R_h \mathbf{u}$, where $R_h \mathbf{u} = \sum_T \mathbf{b}_T \alpha_T(\mathbf{u})$ with \mathbf{b}_T a bubble function and $\alpha_T \in \mathbb{R}^3$ such as

$$\int_T (\mathbf{u} - I_h \mathbf{u}) = 0, \quad \forall T \in \mathcal{T}_h, \quad (25)$$

that is

$$\alpha_T(\mathbf{u}) = \frac{\int_T (\mathbf{u} - \tilde{I}_h \mathbf{u})}{\int_T \mathbf{b}_T} \quad \forall T \in \mathcal{T}_h.$$

Then, (22) can be deduced from (25). Moreover, by using again (25), it is known by means of a duality argument ([9]) that

$$\|\mathbf{u} - I_h \mathbf{u}\|_{H^{-1}} \leq C h |\mathbf{u} - I_h \mathbf{u}|.$$

Now, in order to obtain estimate (24) but changing \tilde{I}_h by I_h it suffices to prove

$$|R_h(\mathbf{u})| \leq C h \|\mathbf{u}\| \quad \text{and} \quad |\nabla R_h \mathbf{u}| \leq C h \|\mathbf{u}\|_{H^2}.$$

Indeed, by using orthogonality of the bubble functions,

$$\begin{aligned} \|R_h(\mathbf{u})\|_{L^2}^2 &= \sum_T \left| \alpha_T(\mathbf{u}) \right|^2 \|\mathbf{b}_T\|_{L^2}^2 = \sum_T \left(\int_T \mathbf{u} - \tilde{I}_h \mathbf{u} \right)^2 \frac{\int_T |\mathbf{b}_T|^2}{\left(\int_T \mathbf{b}_T \right)^2} \\ &\leq C \sum_T |T| \left(\int_T |\mathbf{u} - \tilde{I}_h \mathbf{u}|^2 \right) \frac{|T|}{|T|^2} \leq C \|\mathbf{u} - \tilde{I}_h \mathbf{u}\|_{L^2}^2 \end{aligned}$$

hence $|R_h(\mathbf{u})| \leq C h \|\mathbf{u}\|$, owing to the approximation property $|\mathbf{u} - \tilde{I}_h \mathbf{u}| \leq C h \|\mathbf{u}\|$.

Taking the L^2 -norm of the gradient,

$$\|\nabla R_h(\mathbf{u})\|_{L^2}^2 = \sum_T \left(\int_T \mathbf{u} - \tilde{I}_h \mathbf{u} \right)^2 \frac{\int_T |\nabla \mathbf{b}_T|^2}{\left(\int_T \mathbf{b}_T \right)^2} \leq C \sum_T |T| \left(\int_T |\mathbf{u} - \tilde{I}_h \mathbf{u}|^2 \right) \frac{1}{|T|^2} \leq \frac{C}{h^2} \|\mathbf{u} - \tilde{I}_h \mathbf{u}\|_{L^2}^2$$

hence $\|\nabla R_h(\mathbf{u})\|_{L^2}^2 \leq C h^2 \|\mathbf{u}\|_{H^2}^2$, owing to the approximation property $\|\mathbf{u} - \tilde{I}_h \mathbf{u}\|_{L^2} \leq C h^2 \|\mathbf{u}\|_{H^2}$.

Now, following the equality $\mathbf{u}^{m+1} = \tilde{\mathbf{u}}^{m+1} - k \nabla(p^{m+1} - p^m)$, we define:

$$K_{h,k} \mathbf{u}^{m+1} := I_h \tilde{\mathbf{u}}^{m+1} - k \nabla J_h(p^{m+1} - p^m). \quad (26)$$

Note that $K_{h,k} \mathbf{u}^{m+1} \in \mathbf{Y}_h + \nabla Q_h$. By comparing (26) with the time discrete Algorithm 1:

$$\mathbf{u}^{m+1} - K_{h,k} \mathbf{u}^{m+1} = \tilde{\mathbf{u}}^{m+1} - I_h \tilde{\mathbf{u}}^{m+1} - k \nabla \left((p^{m+1} - J_h p^{m+1}) - (p^m - J_h p^m) \right),$$

hence, using the L^2 approximation property for I_h , the H^1 -stability for J_h and the $H^2 \times H^1$ estimates for $(\tilde{\mathbf{u}}^{m+1}, p^{m+1})$:

$$|\mathbf{u}^{m+1} - K_{h,k} \mathbf{u}^{m+1}| \leq C \left(h^2 \|\tilde{\mathbf{u}}^{m+1}\|_{H^2} + k \|p^{m+1} - p^m\| \right) \leq C(k + h^2) \quad \forall m.$$

The fully discrete scheme is described in Algorithm 2.

Algorithm 2 Fully discrete algorithm

Initialization: Let $(\tilde{\mathbf{u}}_h^0, p_h^0) \in \mathbf{Y}_h \times Q_h$ be an approximation of (\mathbf{u}^0, p^0) . Put $\mathbf{u}_h^0 = \tilde{\mathbf{u}}_h^0$.

Step of time $m + 1$: Let $(\tilde{\mathbf{u}}_h^m, p_h^m) \in \mathbf{Y}_h \times Q_h$ and $\mathbf{u}_h^m \in \mathbf{Y}_h + \nabla Q_h$ be given.

Sub-step 1 : Find $\tilde{\mathbf{u}}_h^{m+1} \in \mathbf{Y}_h$ such that,

$$(S_1)_h^{m+1} \quad \left(\frac{\tilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m}{k}, \mathbf{v}_h \right) + c \left(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h \right) + \left(\nabla \tilde{\mathbf{u}}_h^{m+1}, \nabla \mathbf{v}_h \right) + \left(\nabla p_h^m, \mathbf{v}_h \right) = \left(\mathbf{f}^{m+1}, \mathbf{v}_h \right).$$

Sub-step 2 : Find $p_h^{m+1} \in Q_h$ such that

$$(S_2)_{a,h}^{m+1} \quad \left(k \nabla(p_h^{m+1} - p_h^m), \nabla q_h \right) = \left(\tilde{\mathbf{u}}_h^{m+1}, \nabla q_h \right) \quad \forall q_h \in Q_h.$$

Now, we define $\mathbf{u}_h^{m+1} \in \mathbf{Y}_h + \nabla Q_h$ by

$$(S_2)_{b,h}^{m+1} \quad \mathbf{u}_h^{m+1} = \tilde{\mathbf{u}}_h^{m+1} - k \nabla(p_h^{m+1} - p_h^m).$$

Notice that, adding both sub-steps of Algorithm 2, we obtain:

$$(S_3)_h^{m+1} \quad \left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{k}, \mathbf{v}_h \right) + c \left(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h \right) + \left(\nabla \tilde{\mathbf{u}}_h^{m+1}, \nabla \mathbf{v}_h \right) + \left(\nabla p_h^{m+1}, \mathbf{v}_h \right) = \left(\mathbf{f}^{m+1}, \mathbf{v}_h \right).$$

From $(S_2)_{b,h}^{m+1}$, one has the orthogonality property

$$\left(\mathbf{u}_h^{m+1}, \nabla q_h\right) = 0 \quad \forall q_h \in Q_h. \quad (27)$$

Remark 15 (Segregated version of Algorithm 2) *We introduce the end-of-step velocity \mathbf{u}_h^m only for doing the numerical analysis. For practical implementations, this velocity \mathbf{u}_h^m can be eliminated, rewriting Algorithm 2 as follows:*

Let $(p_h^{m-1}, p_h^m, \tilde{\mathbf{u}}_h^m) \in Q_h \times Q_h \times \mathbf{Y}_h$ be given.

(a) Find $\tilde{\mathbf{u}}_h^{m+1} \in \mathbf{Y}_h$ such that, $\forall \mathbf{v}_h \in \mathbf{Y}_h$:

$$\left(\frac{\tilde{\mathbf{u}}_h^{m+1} - \tilde{\mathbf{u}}_h^m}{k}, \mathbf{v}_h\right) + c(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h) + \left(\nabla \tilde{\mathbf{u}}_h^{m+1}, \nabla \mathbf{v}_h\right) + \left(\nabla(2p_h^m - p_h^{m-1}), \mathbf{v}_h\right) = \left(\mathbf{f}^{m+1}, \mathbf{v}_h\right).$$

(b) Find $p_h^{m+1} \in Q_h$ such that, $\forall q_h \in Q_h$:

$$\left(k \nabla(p_h^{m+1} - p_h^m), \nabla q_h\right) = \left(\tilde{\mathbf{u}}_h^{m+1}, \nabla q_h\right).$$

Then, computations for pressure p_h^{m+1} and velocity $\tilde{\mathbf{u}}_h^{m+1}$ are decoupled. In fact, (a) is a linear convection-diffusion-Dirichlet problem for $\tilde{\mathbf{u}}_h^{m+1}$ (where each component of $\tilde{\mathbf{u}}_h^{m+1}$ is also decoupled from the other ones) and (b) is a Poisson-Neumann problem for p_h^{m+1} . Therefore, Algorithm 2 can be rewritten as a fully decoupled scheme.

Note that, in order to initialize the scheme we have to start with a pressure p_h^{-1} which has not sense. We can avoid it starting from an auxiliary initial step given by either one-step scheme or by the scheme written as Algorithm 2, i.e., given $\tilde{\mathbf{u}}_h^0, p_h^0$ and $\mathbf{u}_h^0 = \tilde{\mathbf{u}}_h^0$, we compute first $\tilde{\mathbf{u}}_h^1$ from $(S_1)_h^1$ and after p_h^1 from $(S_2)_{a,h}^1$.

2.2 Stability and convergence of Algorithm 2

It is easy to extend the results given in the previous Section about the continuous dependence of the projection step of Algorithm 1 to the fully discrete Algorithm 2. Indeed, from $(S_2)_{b,h}^{m+1}$ and the orthogonality property (27), we have

$$|\tilde{\mathbf{u}}_h^{m+1}|^2 = |\mathbf{u}_h^{m+1}|^2 + |k \nabla(p_h^{m+1} - p_h^m)|^2 \quad (28)$$

hence, in particular, $|\mathbf{u}_h^{m+1}| \leq |\tilde{\mathbf{u}}_h^{m+1}|$. From $(S_2)_{a,h}^{m+1}$

$$|k \nabla(p_h^{m+1} - p_h^m)|^2 = (\tilde{\mathbf{u}}_h^{m+1}, k \nabla(p_h^{m+1} - p_h^m)) = (\tilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m, k \nabla(p_h^{m+1} - p_h^m)),$$

hence

$$|\mathbf{u}_h^{m+1} - \tilde{\mathbf{u}}_h^{m+1}| \leq |\tilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m|.$$

Moreover, using the antisymmetric property $c(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{u}}_h^{m+1}, \tilde{\mathbf{u}}_h^{m+1}) = 0$ (see (1)), one can extend the stability and convergence results of Algorithm 1 to the fully discrete Algorithm 2. In particular, for any $r < N$, the following stability estimates hold:

$$\begin{aligned} \|\mathbf{u}_h^{r+1}\|_{l^\infty(L^2)} + \|\tilde{\mathbf{u}}_h^{r+1}\|_{l^\infty(L^2) \cap l^2(H^1)} + \|k \nabla p_h^{r+1}\|_{l^\infty(L^2)} &\leq C, \\ \sum_{m=0}^r |\tilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m|^2 + \sum_{m=0}^r |\mathbf{u}_h^{m+1} - \tilde{\mathbf{u}}_h^{m+1}|^2 &\leq C. \end{aligned} \quad (29)$$

Indeed, by making $\left((S_1)_h^{m+1}, 2k \tilde{\mathbf{u}}_h^{m+1}\right)$, using the fact that

$$2k(\nabla p_h^m, \tilde{\mathbf{u}}_h^{m+1}) = 2(k \nabla p_h^m, k \nabla(p_h^{m+1} - p_h^m)),$$

and the equalities $(a-b)2a = a^2 - b^2 + (a-b)^2$ and $(a-b)2b = a^2 - b^2 - (a-b)^2$, we have

$$\begin{aligned} |\tilde{\mathbf{u}}_h^{m+1}|^2 - |\mathbf{u}_h^m|^2 + |\tilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m|^2 + k \|\tilde{\mathbf{u}}_h^{m+1}\|^2 + |k \nabla p_h^{m+1}|^2 - |k \nabla p_h^m|^2 \\ - |k \nabla(p_h^{m+1} - p_h^m)|^2 \leq k \|\mathbf{f}^{m+1}\|_{H^{-1}}^2 \end{aligned} \quad (30)$$

Adding (28) and (30), the negative term $-|k \nabla(p_h^{m+1} - p_h^m)|^2$ of (30) cancel and we arrive at

$$|\mathbf{u}_h^{m+1}|^2 - |\mathbf{u}_h^m|^2 + |\tilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m|^2 + |k \nabla p_h^{m+1}|^2 - |k \nabla p_h^m|^2 + k \|\tilde{\mathbf{u}}_h^{m+1}\|^2 \leq k \|\mathbf{f}^{m+1}\|_{H^{-1}}^2$$

Now, adding from $m = 0$ to r ($r < N$), we obtain the desired stability estimates (29).

2.3 Problems related to the spatial errors

We will present an error analysis for the fully discrete Algorithm 2 $(\tilde{\mathbf{u}}_h^{m+1}, \mathbf{u}_h^{m+1}, p_h^{m+1})$ as an approximation of the time discrete Algorithm 1 $(\tilde{\mathbf{u}}^{m+1}, \mathbf{u}^{m+1}, p^{m+1})$. Consequently, we define the following errors:

$$\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}, \quad \tilde{\mathbf{e}}_d^{m+1} = \tilde{\mathbf{u}}^{m+1} - \tilde{\mathbf{u}}_h^{m+1}, \quad e_{p,d}^{m+1} = p^{m+1} - p_h^{m+1}.$$

Splitting the discrete part and the interpolation one:

$$\mathbf{e}_d^{m+1} = \mathbf{e}_h^{m+1} + \mathbf{e}_i^{m+1}, \quad \tilde{\mathbf{e}}_d^{m+1} = \tilde{\mathbf{e}}_h^{m+1} + \tilde{\mathbf{e}}_i^{m+1}, \quad e_{p,d}^{m+1} = e_{p,h}^{m+1} + e_{p,i}^{m+1}$$

where \mathbf{e}_i are interpolation errors and \mathbf{e}_h space discrete errors, concretely

$$\begin{aligned} \mathbf{e}_h^{m+1} &= K_{h,k} \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1} \text{ and } \mathbf{e}_i^{m+1} = \mathbf{u}^{m+1} - K_{h,k} \mathbf{u}^{m+1}, \\ \tilde{\mathbf{e}}_h^{m+1} &= I_h \tilde{\mathbf{u}}^{m+1} - \tilde{\mathbf{u}}_h^{m+1} \text{ and } \tilde{\mathbf{e}}_i^{m+1} = \tilde{\mathbf{u}}^{m+1} - I_h \tilde{\mathbf{u}}^{m+1}, \\ e_{p,h}^{m+1} &= J_h p^{m+1} - p_h^{m+1} \text{ and } e_{p,i}^{m+1} = p^{m+1} - J_h p^{m+1}. \end{aligned}$$

Remark 16 From the equalities $\mathbf{u}^{m+1} = \tilde{\mathbf{u}}^{m+1} - k \nabla(p^{m+1} - p^m)$ and $K_{h,k} \mathbf{u}^{m+1} = I_h \tilde{\mathbf{u}}^{m+1} - k \nabla J_h(p^{m+1} - p^m)$, one has

$$\mathbf{e}_i^{m+1} = \tilde{\mathbf{e}}_i^{m+1} - k \nabla(e_{p,i}^{m+1} - e_{p,i}^m). \quad (31)$$

In particular, subtracting $\tilde{\mathbf{e}}_i^{m+1}$ and (31) replacing m for $m-1$, we get

$$\frac{1}{k}(\tilde{\mathbf{e}}_i^{m+1} - \mathbf{e}_i^m) = e_i(\delta_t \tilde{\mathbf{u}}^{m+1}) + \nabla(e_{p,i}^m - e_{p,i}^{m-1}), \quad (32)$$

where $e_i(\delta_t \tilde{\mathbf{u}}^{m+1}) = (\tilde{\mathbf{e}}_i^{m+1} - \tilde{\mathbf{e}}_i^m)/k$. Moreover, owing to the choice of the interpolation operators I_h and J_h , from (31)

$$\left(\mathbf{e}_i^{m+1}, \nabla q_h \right) = \left(\tilde{\mathbf{e}}_i^{m+1}, \nabla q_h \right) - k \left(\nabla(e_{p,i}^{m+1} - e_{p,i}^m), \nabla q_h \right) = 0, \quad \forall q_h \in Q_h. \quad (33)$$

On the other hand, since $\left(\mathbf{u}_h^{m+1}, \nabla q_h \right) = 0 \quad \forall q_h \in Q_h$ and $\left(\mathbf{u}^{m+1}, \nabla q \right) = 0 \quad \forall q \in H^1 \cap L_0^2$, then

$$\left(\mathbf{e}_d^{m+1}, \nabla q_h \right) = 0 \quad \forall q_h \in Q_h. \quad (34)$$

Finally, from (33) and (34), we arrive at

$$\left(\mathbf{e}_h^{m+1}, \nabla q_h \right) = 0 \quad \forall q_h \in Q_h.$$

By comparing $(S_1)^{m+1}, (S_2)^{m+1}$ and $(S_1)_h^{m+1}, (S_2)_{b,h}^{m+1}$, we have the following problems satisfied by the spatial errors $\tilde{\mathbf{e}}_d^{m+1}$ and $(\mathbf{e}_d^{m+1}, e_{p,d}^{m+1})$ respectively:

$$\frac{1}{k} \left(\tilde{\mathbf{e}}_d^{m+1} - \mathbf{e}_d^m, \mathbf{v}_h \right) + \left(\nabla \tilde{\mathbf{e}}_d^{m+1}, \nabla \mathbf{v}_h \right) + \left(\nabla e_{p,d}^m, \mathbf{v}_h \right) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{Y}_h,$$

and

$$\mathbf{e}_d^{m+1} = \tilde{\mathbf{e}}_d^{m+1} - k \nabla(e_{p,d}^{m+1} - e_{p,d}^m),$$

where

$$\mathbf{NL}_h^{m+1}(\mathbf{v}_h) = c \left(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h \right) - c \left(\tilde{\mathbf{u}}^m, \tilde{\mathbf{u}}^{m+1}, \mathbf{v}_h \right) = -c \left(\tilde{\mathbf{e}}_d^m, \tilde{\mathbf{u}}^{m+1}, \mathbf{v}_h \right) - c \left(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{e}}_d^{m+1}, \mathbf{v}_h \right).$$

By splitting the error in the discrete and the interpolation parts and using (31) and (32),

$$(E_1)_h^{m+1} \quad \left\{ \begin{array}{l} \frac{1}{k} \left(\tilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m, \mathbf{v}_h \right) + \left(\nabla \tilde{\mathbf{e}}_h^{m+1}, \nabla \mathbf{v}_h \right) + \left(\nabla e_{p,h}^m, \mathbf{v}_h \right) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ - \left(e_i(\delta_t \tilde{\mathbf{u}}^{m+1}), \mathbf{v}_h \right) - \left(\nabla \tilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h \right) - \left(\nabla(2e_{p,i}^m - e_{p,i}^{m-1}), \mathbf{v}_h \right), \quad \forall \mathbf{v}_h \in \mathbf{Y}_h \end{array} \right.$$

$$(E_2)_h^{m+1} \quad \mathbf{e}_h^{m+1} = \tilde{\mathbf{e}}_h^{m+1} - k \nabla(e_{p,h}^{m+1} - e_{p,h}^m).$$

Finally, adding $(E_1)_h^{m+1}$ and $(E_2)_h^{m+1}$,

$$(E_3)_h^{m+1} \quad \left\{ \begin{array}{l} \frac{1}{k} \left(\mathbf{e}_h^{m+1} - \mathbf{e}_h^m, \mathbf{v}_h \right) + \left(\nabla \tilde{\mathbf{e}}_h^{m+1}, \nabla \mathbf{v}_h \right) + \left(\nabla e_{p,h}^{m+1}, \mathbf{v}_h \right) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ - \left(e_i(\delta_t \tilde{\mathbf{u}}^{m+1}), \mathbf{v}_h \right) - \left(\nabla \tilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h \right) - \left(\nabla(2e_{p,i}^m - e_{p,i}^{m-1}), \mathbf{v}_h \right), \quad \forall \mathbf{v}_h \in \mathbf{Y}_h. \end{array} \right.$$

2.4 $O(h)$ error estimates for $\tilde{\mathbf{e}}_h^{m+1}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and for \mathbf{e}_h^{m+1} in $l^\infty(\mathbf{L}^2)$

Theorem 17 *We assume hypotheses of Theorem 7 and the initial approximation*

$$|\mathbf{e}_h^0| + |k \nabla e_{p,h}^0| \leq C h.$$

Then, the following error estimates hold

$$\|\tilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)}^2 + \|\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)}^2 + \|k \nabla e_{p,h}^{m+1}\|_{l^\infty(\mathbf{L}^2)}^2 \leq C h^2, \quad (35)$$

$$\|\tilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m\|_{l^2(\mathbf{L}^2)}^2 \leq C k h^2. \quad (36)$$

Remark 18 *By using the $O(k)$ accuracy for the time discrete Algorithm 1, we arrive at the following optimal order for the total error of the velocity:*

$$\|\mathbf{u}(t_{m+1}) - \tilde{\mathbf{u}}_h^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} \leq C(k + h).$$

Proof: By making $((E_1)_h^{m+1}, 2k\tilde{\mathbf{e}}_h^{m+1})$ and using the equalities

$$(\nabla e_{p,h}^m, \mathbf{e}_h^{m+1}) = 0,$$

$$2k(\nabla e_{p,h}^m, \tilde{\mathbf{e}}_h^{m+1}) = 2(k \nabla e_{p,h}^m, k \nabla(e_{p,h}^{m+1} - e_{p,h}^m)) = |k \nabla e_{p,h}^{m+1}|^2 - |k \nabla e_{p,h}^m|^2 - |k \nabla(e_{p,h}^{m+1} - e_{p,h}^m)|^2,$$

and the L^2 -orthogonality property

$$|\tilde{\mathbf{e}}_h^{m+1}|^2 = |\mathbf{e}_h^{m+1}|^2 + |k \nabla(e_{p,h}^{m+1} - e_{p,h}^m)|^2, \quad (37)$$

we arrive at

$$\begin{aligned} & |\mathbf{e}_h^{m+1}|^2 - |\mathbf{e}_h^m|^2 + |\tilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m|^2 + 2k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + |k \nabla e_{p,h}^{m+1}|^2 - |k \nabla e_{p,h}^m|^2 \\ &= -2k(e_i(\delta_t \tilde{\mathbf{u}}^{m+1}), \tilde{\mathbf{e}}_h^{m+1}) - 2k(\nabla \tilde{\mathbf{e}}_i^{m+1}, \nabla \tilde{\mathbf{e}}_h^{m+1}) - 2k(\nabla(2e_{p,i}^m - e_{p,i}^{m-1}), \tilde{\mathbf{e}}_h^{m+1}) \\ &+ 2kc(\tilde{\mathbf{e}}_h^m, \tilde{\mathbf{u}}^{m+1}, \tilde{\mathbf{e}}_h^{m+1}) + 2kc(\tilde{\mathbf{e}}_i^m, \tilde{\mathbf{u}}^{m+1}, \tilde{\mathbf{e}}_h^{m+1}) - 2kc(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{e}}_h^{m+1}, \tilde{\mathbf{e}}_h^{m+1}) \\ &- 2kc(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{e}}_i^{m+1}, \tilde{\mathbf{e}}_h^{m+1}) := \sum_{i=1}^7 I_i \end{aligned} \quad (38)$$

We bound the RHS of (38) as follows (using Remark 12):

$$I_1 \leq \varepsilon k |\tilde{\mathbf{e}}_h^{m+1}|^2 + C k |e_i(\delta_t \tilde{\mathbf{u}}^{m+1})|^2 \leq \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + C h^2 k \|\delta_t \tilde{\mathbf{u}}^{m+1}\|^2$$

$$I_2 \leq \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + C h^2 k \|\tilde{\mathbf{u}}^{m+1}\|_{H^2}^2 \leq \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + C k h^2$$

$$I_3 = 2k((2e_{p,i}^m - e_{p,i}^{m-1}), \nabla \cdot \tilde{\mathbf{e}}_h^{m+1}) \leq \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + C k h^2 (\|p^m\|^2 + \|p^{m-1}\|^2) \leq \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + C k h^2$$

With respect to the nonlinear terms,

$$I_4 = 2kc(\tilde{\mathbf{e}}_h^m, \tilde{\mathbf{u}}^{m+1}, \tilde{\mathbf{e}}_h^{m+1}) \leq C k |\tilde{\mathbf{e}}_h^m| \|\tilde{\mathbf{u}}^{m+1}\|_{W^{1,3} \cap L^\infty} \|\tilde{\mathbf{e}}_h^{m+1}\| \leq \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + C k |\tilde{\mathbf{e}}_h^m|^2$$

$$\leq \varepsilon k \|\tilde{\mathbf{e}}^{m+1}\|^2 + C k \left(|\mathbf{e}_h^m|^2 + 2|k \nabla e_{p,h}^m|^2 + 2|k \nabla e_{p,h}^{m-1}|^2 \right)$$

(here, (37) has been used),

$$\begin{aligned} I_5 &= 2 k c \left(\tilde{\mathbf{e}}_i^m, \tilde{\mathbf{u}}^{m+1}, \tilde{\mathbf{e}}_h^{m+1} \right) \leq \varepsilon k \|\tilde{\mathbf{e}}^{m+1}\|^2 + C k |\tilde{\mathbf{e}}_i^m|^2 \leq \varepsilon k \|\tilde{\mathbf{e}}^{m+1}\|^2 + C h^4 k \|\tilde{\mathbf{u}}^m\|_{H^2}^2 \\ &\leq \varepsilon k \|\tilde{\mathbf{e}}^{m+1}\|^2 + C k h^4, \end{aligned}$$

$$I_6 = 2 k c \left(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{e}}_h^{m+1}, \tilde{\mathbf{e}}_h^{m+1} \right) = 0,$$

$$\begin{aligned} I_7 &= 2 k c \left(\tilde{\mathbf{u}}_h^m, \tilde{\mathbf{e}}_i^{m+1}, \tilde{\mathbf{e}}_h^{m+1} \right) \leq C k \|\tilde{\mathbf{u}}_h^m\|^2 \|\tilde{\mathbf{e}}_i^{m+1}\|_{L^3}^2 + \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 \\ &\leq C k \|\tilde{\mathbf{u}}_h^m\|^2 |\tilde{\mathbf{e}}_i^{m+1}| \|\tilde{\mathbf{e}}_i^{m+1}\| + \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 \\ &\leq C k h^3 \|\tilde{\mathbf{u}}_h^m\|^2 \|\tilde{\mathbf{u}}^{m+1}\|_{H^2}^2 + \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 \\ &\leq C k h^3 \|\tilde{\mathbf{u}}_h^m\|^2 + \varepsilon k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 \end{aligned}$$

Then, using these bounds in (38) we obtain

$$\begin{aligned} &|\mathbf{e}_h^{m+1}|^2 - |\mathbf{e}_h^m|^2 + |k \nabla e_{p,h}^{m+1}|^2 - |k \nabla e_{p,h}^m|^2 + |\tilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m|^2 + k \|\tilde{\mathbf{e}}_h^{m+1}\|^2 \\ &\leq C k \left(|\mathbf{e}_h^m|^2 + |k \nabla e_{p,h}^m|^2 + |k \nabla e_{p,h}^{m-1}|^2 \right) + C k h^2 + C k h^3 \|\tilde{\mathbf{u}}_h^m\|^2 + C k h^2 \|\delta_t \tilde{\mathbf{u}}^{m+1}\|^2. \end{aligned} \quad (39)$$

Finally, by adding (39) from $m = 0$ to r (with any $r < M$), and using that $k \sum \|\tilde{\mathbf{u}}_h^m\|^2 \leq C$ and Theorem 9, the discrete Gromwall's Lemma yields to

$$|\mathbf{e}_h^{r+1}|^2 + |k \nabla e_{p,h}^{r+1}|^2 + \sum_{m=0}^r |\tilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m|^2 + k \sum_{m=0}^r \|\tilde{\mathbf{e}}_h^{m+1}\|^2 \leq C \left(|\mathbf{e}_h^0|^2 + |k \nabla e_{p,h}^0|^2 + h^2 \right)$$

hence the estimates (35)-(36) hold. \blacksquare

Theorem 17 and the inverse inequality $\|\mathbf{u}_h\| \leq C h^{-1} |\mathbf{u}_h|$ for each $\mathbf{u}_h \in \mathbf{Y}_h$, imply the uniform estimate

$$\|\tilde{\mathbf{e}}_h^{m+1}\| \leq \frac{C}{h} |\tilde{\mathbf{e}}_h^{m+1}| \leq C. \quad (40)$$

2.5 $O(h)$ for $\delta_t \mathbf{e}_h^{m+1}$ in $l^\infty(\mathbf{L}^2)$, $\delta_t \tilde{\mathbf{e}}_h^{m+1}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and $(\tilde{\mathbf{e}}_h^{m+1}, e_{p,d}^{m+1})$ in $l^\infty(\mathbf{H}^1 \times L^2)$

By making $\delta_t(E_1)_h^{m+1}$ and $\delta_t(E_2)_h^{m+1}$, one arrives at ($\forall m \geq 1$):

$$\frac{1}{k} \left(\delta_t \tilde{\mathbf{e}}_d^{m+1} - \delta_t \mathbf{e}_d^m, \mathbf{v}_h \right) + \left(\nabla \delta_t \tilde{\mathbf{e}}_d^{m+1}, \nabla \mathbf{v}_h \right) - \left(\delta_t \nabla e_{p,d}^m, \mathbf{v}_h \right) = \delta_t \mathbf{N} \mathbf{L}_h^{m+1}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{Y}_h$$

and

$$\delta_t \mathbf{e}_d^{m+1} = \delta_t \tilde{\mathbf{e}}_d^{m+1} - k \nabla (\delta_t e_{p,d}^{m+1} - \delta_t e_{p,d}^m)$$

where

$$\delta_t \mathbf{NL}_h^{m+1}(\mathbf{v}_h) = c\left(\delta_t \tilde{\mathbf{e}}_d^m, \tilde{\mathbf{u}}^{m+1}, \mathbf{v}_h\right) + c\left(\delta_t \tilde{\mathbf{u}}_h^m, \tilde{\mathbf{e}}_d^{m+1}, \mathbf{v}_h\right) + c\left(\tilde{\mathbf{e}}_d^{m-1}, \delta_t \tilde{\mathbf{u}}^{m+1}, \mathbf{v}_h\right) + c\left(\tilde{\mathbf{u}}_h^{m-1}, \delta_t \tilde{\mathbf{e}}_d^{m+1}, \mathbf{v}_h\right).$$

On the other hand, the following L^2 -orthogonality property holds:

$$\left(\delta_t \mathbf{e}_d^{m+1}, \nabla q_h\right) = 0, \quad \forall q_h \in Q_h.$$

Consequently, for each $\mathbf{v}_h \in \mathbf{Y}_h$, one has

$$(D_1)_h^{m+1} \quad \begin{cases} \frac{1}{k} \left(\delta_t \tilde{\mathbf{e}}_h^{m+1} - \delta_t \mathbf{e}_h^m, \mathbf{v}_h \right) + \left(\nabla \delta_t \tilde{\mathbf{e}}_h^{m+1}, \nabla \mathbf{v}_h \right) + \left(\nabla \delta_t e_{p,h}^m, \mathbf{v}_h \right) = \delta_t \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ - \left(\mathbf{e}_i(\delta_t \delta_t \tilde{\mathbf{u}}^{m+1}), \mathbf{v}_h \right) - \left(\nabla \delta_t \tilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h \right) - \left(\nabla (2 \delta_t e_{p,i}^m - \delta_t e_{p,i}^{m-1}), \mathbf{v}_h \right), \end{cases}$$

$$(D_2)_h^{m+1} \quad \delta_t \mathbf{e}_h^{m+1} = \delta_t \tilde{\mathbf{e}}_h^{m+1} - k \nabla (\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m),$$

and the following discrete L^2 -orthogonality property:

$$\left(\delta_t \mathbf{e}_h^{m+1}, \nabla q_h\right) = 0, \quad \forall q_h \in Q_h. \quad (41)$$

In the last two equalities, some properties of the interpolation operators have been used.

Theorem 19 *Under the hypotheses of Theorems 9 and 17, assuming the following approximation for the first step of Algorithm 2*

$$|\delta_t \mathbf{e}_h^1| + |k \nabla \delta_t e_{p,h}^1| \leq C h, \quad k \|\delta_t \tilde{\mathbf{e}}_h^1\|^2 \leq C h^2 \quad (42)$$

then

$$\|\delta_t \mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)} + \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} + \|k \delta_t \nabla e_{p,h}^{m+1}\|_{l^\infty(\mathbf{L}^2)} \leq C h. \quad (43)$$

Proof: Since the initial estimate $|\delta_t \mathbf{e}_h^1| + |k \nabla \delta_t e_{p,h}^1| \leq C h$ is assumed, it suffices to prove (43) for each $m \geq 1$.

By adding $(D_1)_h^{m+1}$ multiplied by $2k \delta_t \tilde{\mathbf{e}}_h^{m+1} \in \mathbf{Y}_h$, where the pressure term is writing as

$$\begin{aligned} 2k \left(\nabla \delta_t e_{p,h}^m, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) &= 2k \left(\nabla \delta_t e_{p,h}^m, k \nabla (\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m) \right) \\ &= |k \nabla \delta_t e_{p,h}^{m+1}|^2 - |k \nabla \delta_t e_{p,h}^m|^2 - |k \nabla (\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m)|^2 \end{aligned}$$

(here $\left(\nabla \delta_t e_{p,h}^m, \delta_t \mathbf{e}_h^{m+1}\right) = 0$ has been used), and the equality

$$|\delta_t \tilde{\mathbf{e}}_h^{m+1}|^2 = |\delta_t \mathbf{e}_h^{m+1}|^2 + |k \nabla (\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m)|^2$$

(which is deduced from $(D_2)_h^{m+1}$ and the discrete L^2 -orthogonality (41)), one has

$$\begin{aligned} &|\delta_t \mathbf{e}_h^{m+1}|^2 - |\delta_t \mathbf{e}_h^m|^2 + |\delta_t \tilde{\mathbf{e}}_h^{m+1} - \delta_t \mathbf{e}_h^m|^2 + 2k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + |k \nabla \delta_t e_{p,h}^{m+1}|^2 \\ &- |k \nabla \delta_t e_{p,h}^m|^2 = -2k \left(\mathbf{e}_i(\delta_t \delta_t \tilde{\mathbf{u}}^{m+1}), \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) - 2k \left(\nabla \delta_t \tilde{\mathbf{e}}_i^{m+1}, \nabla \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) \\ &- 2k \left(\nabla (2 \delta_t e_{p,i}^m - \delta_t e_{p,i}^{m-1}), \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) + 2 \delta_t \mathbf{NL}_h^{m+1}(\delta_t \tilde{\mathbf{e}}_h^{m+1}) := I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (44)$$

We bound the RHS of (44) as:

$$I_1 \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + C k h^2 |\delta_t \delta_t \tilde{\mathbf{u}}^{m+1}|^2$$

(here the hypothesis (23) on the $O(h)$ -approximation of I_h in the \mathbf{H}^{-1} -norm has been used),

$$I_2 \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + C k h^2 \|\delta_t \tilde{\mathbf{u}}^{m+1}\|_{\mathbf{H}^2}^2$$

$$I_3 \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + C k h^2 (\|\delta_t p^m\|^2 + \|\delta_t p^{m-1}\|^2).$$

The nonlinear terms, for $m \geq 1$, are treated as follows:

$$\begin{aligned} I_4 &= 2k c \left(\delta_t \tilde{\mathbf{e}}_d^m, \tilde{\mathbf{u}}^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) + 2k c \left(\delta_t \tilde{\mathbf{u}}_h^m, \tilde{\mathbf{e}}_d^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) \\ &+ 2k c \left(\tilde{\mathbf{e}}_d^{m-1}, \delta_t \tilde{\mathbf{u}}^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) + 2k c \left(\tilde{\mathbf{u}}_h^{m-1}, \delta_t \tilde{\mathbf{e}}_d^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) := \sum_{i=1}^4 J_i \end{aligned}$$

We bound each J_i -term as follows:

$$J_1 = 2k c \left(\delta_t \tilde{\mathbf{e}}_h^m, \tilde{\mathbf{u}}^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) + 2k c \left(\delta_t \tilde{\mathbf{e}}_i^m, \tilde{\mathbf{u}}^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) := J_{11} + J_{12}$$

$$J_{11} \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + C k \|\tilde{\mathbf{u}}^{m+1}\|_{W^{1,3} \cap L^\infty}^2 |\delta_t \tilde{\mathbf{e}}_h^m|^2$$

$$\leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + C k \left(|\delta_t \tilde{\mathbf{e}}_h^m|^2 + 2(|k \nabla \delta_t e_p^m|^2 + |k \nabla \delta_t e_p^{m-1}|^2) \right),$$

$$J_{12} \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + C k \|\tilde{\mathbf{u}}^{m+1}\|^2 |\delta_t \tilde{\mathbf{e}}_i^m|^2 \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + C k h^2 \|\delta_t \tilde{\mathbf{u}}^m\|^2,$$

$$J_2 = 2k c \left(\delta_t \tilde{\mathbf{e}}_h^m, \tilde{\mathbf{e}}_d^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) + 2k c \left(\delta_t I_h \tilde{\mathbf{u}}^m, \tilde{\mathbf{e}}_h^{m+1} + \tilde{\mathbf{e}}_i^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right)$$

$$\leq \varepsilon k \left(\|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + \|\delta_t \tilde{\mathbf{e}}_h^m\|^2 \right) + C k \|\tilde{\mathbf{e}}_d^{m+1}\|^2 |\delta_t \tilde{\mathbf{e}}_h^m|^2 + C k \|I_h \delta_t \tilde{\mathbf{u}}^m\|^2 \left(\|\tilde{\mathbf{e}}_i^m\|^2 + \|\tilde{\mathbf{e}}_h^m\|^2 \right)$$

$$\leq \varepsilon k \left(\|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + \|\delta_t \tilde{\mathbf{e}}_h^m\|^2 \right) + C k |\delta_t \tilde{\mathbf{e}}_h^m|^2 + C k \|\delta_t \tilde{\mathbf{u}}^m\|^2 \left(h^2 \|\tilde{\mathbf{u}}^m\|_{\mathbf{H}^2}^2 + \|\tilde{\mathbf{e}}_h^m\|^2 \right)$$

$$\leq \varepsilon k \left(\|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + \|\delta_t \tilde{\mathbf{e}}_h^m\|^2 \right) + C k \left(|\delta_t \tilde{\mathbf{e}}_h^m|^2 + 2(|k \nabla \delta_t e_p^m|^2 + |k \nabla \delta_t e_p^{m-1}|^2) \right) + C k h^2 + C k \|\tilde{\mathbf{e}}_h^m\|^2$$

(in the last inequality we use $\|\tilde{\mathbf{e}}_d^{m+1}\| \leq C$, due to (40) and $\|\tilde{\mathbf{u}}^m\|_{H^2} \leq C$),

$$J_3 = 2k c \left(\tilde{\mathbf{e}}_h^{m-1}, \delta_t \tilde{\mathbf{u}}^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) + 2k c \left(\tilde{\mathbf{e}}_i^{m-1}, \delta_t \tilde{\mathbf{u}}^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) := J_{31} + J_{32}$$

$$J_{31} \leq C k \|\tilde{\mathbf{e}}_h^{m-1}\| \|\delta_t \tilde{\mathbf{u}}^{m+1}\|_{L^3} \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\| \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1/2}\|^2 + C k \|\tilde{\mathbf{e}}_h^{m-1}\|^2,$$

$$J_{32} \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + C k \|\tilde{\mathbf{e}}_i^{m-1}\|^2 \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1/2}\|^2 + C k \|\delta_t \tilde{\mathbf{u}}^{m+1}\|_{H^2}^2 \|\tilde{\mathbf{e}}_i^{m-1}\|^2,$$

$$J_4 = 2k c \left(\tilde{\mathbf{u}}_h^{m-1}, \delta_t \tilde{\mathbf{e}}_h^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) + 2k c \left(\tilde{\mathbf{u}}_h^{m-1}, \delta_t \tilde{\mathbf{e}}_i^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) := J_{41} + J_{42}$$

$$J_{41} = 0,$$

$$J_{42} \leq C k \|\tilde{\mathbf{u}}_h^{m-1}\| \|\delta_t \tilde{\mathbf{e}}_i^{m+1}\|_{L^3} \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\| \leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + C k \|\tilde{\mathbf{u}}_h^{m-1}\|^2 h^3 \|\delta_t \tilde{\mathbf{u}}^{m+1}\|_{H^2}^2$$

(here, we use $\|\delta_t \tilde{\mathbf{e}}_i^{m+1}\|_{L^3} \leq C |\delta_t \tilde{\mathbf{e}}_i^{m+1}|^{1/2} \|\delta_t \tilde{\mathbf{e}}_i^{m+1}\|^{1/2} \leq C h^{3/2} \|\delta_t \tilde{\mathbf{u}}^{m+1}\|_{H^2}$).

By applying these estimates in (44) for a small enough ε , we obtain

$$\begin{aligned} & |\delta_t \mathbf{e}_h^{m+1}|^2 - |\delta_t \mathbf{e}_h^m|^2 + |\delta_t \tilde{\mathbf{e}}_h^{m+1} - \delta_t \mathbf{e}_h^m|^2 + k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + |k \nabla \delta_t e_{p,h}^{m+1}|^2 - |k \nabla \delta_t e_{p,h}^m|^2 \\ & \leq C k \left(|\delta_t \mathbf{e}_h^m|^2 + 2(|k \nabla \delta_t e_p^m|^2 + |k \nabla \delta_t e_p^{m-1}|^2) \right) + C k h^2 + \frac{k}{2} \|\delta_t \tilde{\mathbf{e}}^m\|^2 \\ & + C k h^2 \left(|\delta_t \delta_t \tilde{\mathbf{u}}^{m+1}|^2 + \|\delta_t \tilde{\mathbf{u}}^{m+1}\|_{H^2}^2 \right) + C k \|\tilde{\mathbf{e}}_h^m\|^2. \end{aligned}$$

Therefore, by adding from $m = 1$ to r (with any $r < M$), taking into account (20), Lemma 13 and Theorem 17, the discrete Gromwall's Lemma can be applied, yielding to

$$\begin{aligned} & |\delta_t \mathbf{e}_h^{r+1}|^2 + |k \nabla \delta_t e_{p,h}^{r+1}|^2 + \sum_{m=1}^r |\delta_t \tilde{\mathbf{e}}_h^{m+1} - \delta_t \mathbf{e}_h^m|^2 + \frac{k}{2} \sum_{m=1}^r \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 \\ & \leq C \left(|\delta_t \mathbf{e}_h^1|^2 + \frac{k}{2} \|\delta_t \tilde{\mathbf{e}}^1\|^2 + |k \nabla \delta_t e_{p,h}^1|^2 + h^2 \right), \end{aligned}$$

hence (43) holds by using the hypotheses on the first step (42). \blacksquare

Corollary 20 *Assuming hypotheses of Theorem 19, the following error estimates hold*

$$\|\tilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(H^1)} \leq C h \quad \text{and} \quad \|e_{p,h}^{m+1}\|_{l^\infty(L^2)} \leq C h.$$

Proof: We divide the proof into three steps:

Step 1. To obtain

$$\|e_{p,h}^{m+1}\|_{l^2(L^2)} \leq C h. \quad (45)$$

Arguing as in the time discrete Algorithm 1, from the discrete inf-sup condition applied to $(E_3)_h^{m+1}$ and the estimates $\|\tilde{\mathbf{e}}_h^{m+1}\|_{l^2(H^1)} \leq C h$ and $\|\delta_t \mathbf{e}_h^{m+1}\|_{l^\infty(L^2)} \leq C h$, we have (45).

Step 2. To prove

$$\|\tilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(H^1)} \leq C h. \quad (46)$$

By multiplying $(E_3)_h^{m+1}$ by $2 k \delta_t \tilde{\mathbf{e}}_h^{m+1}$:

$$\begin{aligned} & |\nabla \tilde{\mathbf{e}}_h^{m+1}|^2 - |\nabla \tilde{\mathbf{e}}_h^m|^2 + |\nabla \tilde{\mathbf{e}}_h^{m+1} - \nabla \tilde{\mathbf{e}}_h^m|^2 = -2 k \left(\nabla e_{p,h}^{m+1} + \delta_t \mathbf{e}_h^{m+1} + \delta_t \mathbf{e}_i^{m+1}, \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) \\ & - 2 k \left(\nabla \tilde{\mathbf{e}}_i^{m+1}, \nabla \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) - 2 k \left(\nabla (2 e_{p,i}^m - e_{p,i}^{m-1}), \delta_t \tilde{\mathbf{e}}_h^{m+1} \right) + 2 k \mathbf{NL}_h^{m+1}(\delta_t \tilde{\mathbf{e}}_h^{m+1}). \end{aligned}$$

Then, we obtain

$$\begin{aligned} & |\nabla \tilde{\mathbf{e}}_h^{m+1}|^2 - |\nabla \tilde{\mathbf{e}}_h^m|^2 + |\nabla \tilde{\mathbf{e}}_h^{m+1} - \nabla \tilde{\mathbf{e}}_h^m|^2 \leq 2 k |e_{p,h}^{m+1}|^2 \\ & + C k |\nabla \cdot \delta_t \tilde{\mathbf{e}}_h^{m+1}|^2 + C k |\delta_t \mathbf{e}_h^{m+1}|^2 + C k |\delta_t \tilde{\mathbf{e}}_h^{m+1}|^2 \\ & + 2 k |e_i(\delta_t \mathbf{u}^{m+1})|^2 + 2 k \|\tilde{\mathbf{e}}_i^{m+1}\|^2 + C k (|e_{p,i}^m|^2 + |e_{p,i}^{m-1}|^2) + 2 k \mathbf{NL}_h^{m+1}(\delta_t \tilde{\mathbf{e}}_h^{m+1}) \end{aligned}$$

$$\begin{aligned} &\leq 2k |e_{p,h}^{m+1}|^2 + Ck |\nabla \cdot \delta_t \tilde{\mathbf{e}}_h^{m+1}|^2 + Ck |\delta_t \mathbf{e}_h^{m+1}|^2 + Ck |\delta_t \tilde{\mathbf{e}}_h^{m+1}|^2 \\ &+ Ck h^2 \|\delta_t \mathbf{u}^{m+1}\|^2 + Ck h^2 \|\tilde{\mathbf{u}}^{m+1}\|_{H^2}^2 + Ck h^2 (\|p^m\|^2 + \|p^{m-1}\|^2) + 2k \mathbf{NL}_h^{m+1}(\delta_t \tilde{\mathbf{e}}_h^{m+1}). \end{aligned}$$

Taking into account (40), we bound the last term of the RHS as follows,

$$\begin{aligned} 2k \mathbf{NL}_h^{m+1}(\delta_t \tilde{\mathbf{e}}_h^{m+1}) &\leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + Ck \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + Ck \|\tilde{\mathbf{e}}_h^m\|^2 + Ck \|\tilde{\mathbf{e}}_i^{m+1}\|^2 + Ck \|\tilde{\mathbf{e}}_i^m\|^2 \\ &\leq \varepsilon k \|\delta_t \tilde{\mathbf{e}}_h^{m+1}\|^2 + Ck \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + Ck \|\tilde{\mathbf{e}}_h^m\|^2 + Ck h^2 \end{aligned}$$

hence, we arrive at

$$\begin{aligned} &|\nabla \tilde{\mathbf{e}}_h^{m+1}|^2 - |\nabla \tilde{\mathbf{e}}_h^m|^2 + |\nabla \tilde{\mathbf{e}}_h^{m+1} - \nabla \tilde{\mathbf{e}}_h^m|^2 \leq k |e_{p,h}^{m+1}|^2 + k |\nabla \cdot \delta_t \tilde{\mathbf{e}}_h^{m+1}|^2 + Ck |\delta_t \mathbf{e}_h^{m+1}|^2 \\ &+ k |\nabla \delta_t \tilde{\mathbf{e}}_h^{m+1}|^2 + Ck |\delta_t \tilde{\mathbf{e}}_h^{m+1}|^2 + Ck \|\tilde{\mathbf{e}}_h^{m+1}\|^2 + Ck \|\tilde{\mathbf{e}}_h^m\|^2 + Ck h^2 + Ck h^2 \|\delta_t \mathbf{u}^{m+1}\|^2 \end{aligned}$$

Adding from $m = 0$ to r ,

$$|\nabla \tilde{\mathbf{e}}_h^{r+1}|^2 \leq |\nabla \tilde{\mathbf{e}}_h^0|^2 + Ck \sum_{m=0}^r |e_{p,h}^{m+1}|^2 + Ck \sum_{m=0}^r |\nabla \delta_t \tilde{\mathbf{e}}_h^{m+1}|^2 + Ck \sum_{m=0}^r |\delta_t \mathbf{e}_h^{m+1}|^2 + Ck h^2.$$

Then, by applying (45) and the estimates obtained in Theorems 17 and 19, we obtain (46).

Step 3. To obtain $\|e_{p,h}^{m+1}\|_{l^\infty(L^2)} \leq Ch$.

Finally, by using again the discrete inf-sup condition (21) and taking into account (46), one has $\|e_{p,h}^{m+1}\|_{l^\infty(L^2)} \leq Ch$ and the proof is finished. \blacksquare

Remark 21 By combining Theorem 10 and Corollary 20, the following error estimate for the total error holds

$$\|\mathbf{u}(t_{m+1}) - \tilde{\mathbf{u}}_h^{m+1}\|_{l^\infty(H^1)} + \|p(t_{m+1}) - p_h^{m+1}\|_{l^\infty(L^2)} \leq C(k + h).$$

3 Numerical Simulations

We consider the FE approximation $P_2 \times P_1$ related to a structured mesh of the domain $\Omega = (0, 1)^2 \subset \mathbb{R}^2$.

The numerical results have been obtained using the software FreeFem++ ([8, 20]), and show first order accurate in time for velocity and pressure of the segregated version of Algorithm 2 given in Remark 15. These results are agree to Remark 21.

In fact, we present some numerical error orders in time for velocity $\mathbf{u} = (u_1, u_2)$ and pressure p using the following exact solution for (P) :

$$\mathbf{u} = e^{-t} \begin{pmatrix} (\cos(2\pi x) - 1) \sin(2\pi y) \\ -(\cos(2\pi y) - 1) \sin(2\pi x) \end{pmatrix} \quad \text{and} \quad p = 2\pi e^{-t} (\sin(2\pi x) + \sin(2\pi y)).$$

We take $\nu = 1$ and adjust the force \mathbf{f} to enforce this exact solution.

Note that $\nabla \cdot \mathbf{u} = 0$ in Ω , $\mathbf{u}|_{\partial\Omega} = 0$ and $\int_{\Omega} p = 0$. On the other hand, we have choice this regular exact solution such that $\nabla p \cdot \mathbf{n} \neq 0$ on the boundary $\partial\Omega$, in order to measure the effect of the numerical boundary condition $\nabla(p^{n+1} - p^n) \cdot \mathbf{n} = 0$ on $\partial\Omega$. We approach numerically the order in time for the segregated version of Algorithm 2 given in Remark 15, comparing to other current first order splitting schemes like, rotational pressure-correction, consistent splitting and penalty-projection schemes, also implemented in they segregated form.

Some numerical analysis results and computational simulations can be seen in [14] and [15] for the rotational pressure-correction projection scheme, in [13], [15] and [29] for the consistent splitting scheme and in [1], [2] and [7] for the penalty-projection scheme.

Concretely, let $\mathbf{u}_h^m \in \mathbf{Y}_h$, $\Pi_h(\nabla \cdot \mathbf{u}^m) \in Q_h$ and $p_h^{m-1}, p_h^m \in Q_h$ be given, where Π_h is the $L^2(\Omega)$ -projector operator onto the discrete pressure space Q_h , the implemented segregated schemes are:

- The *rotational pressure-correction* scheme:

- (a) Find $\mathbf{u}_h^{m+1} \in \mathbf{Y}_h$ such that $\forall \mathbf{v}_h \in \mathbf{Y}_h$,

$$\left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{k}, \mathbf{v}_h \right) + c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1}, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^{m+1}, \nabla \mathbf{v}_h) - (q_h^m, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h).$$

where

$$q_h^m = 2p_h^m - p_h^{m-1} + \nu \Pi_h(\nabla \cdot \mathbf{u}^m) \in Q_h. \quad (47)$$

- (b) Compute $\Pi_h(\nabla \cdot \mathbf{u}_h^{m+1})$.

- (c) Find $p_h^{m+1} \in Q_h$ such that

$$k(\nabla(p_h^{m+1} - p_h^m + \nu \Pi_h(\nabla \cdot \mathbf{u}_h^{m+1})), \nabla q_h) = -(\nabla \cdot \mathbf{u}_h^{m+1}, q_h) \quad \forall q_h \in Q_h \quad (48)$$

- The *consistent splitting* scheme:

- (a) Find $\mathbf{u}_h^{m+1} \in \mathbf{Y}_h$ such that $\forall \mathbf{v}_h \in \mathbf{Y}_h$,

$$\left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{k}, \mathbf{v}_h \right) + c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1}, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^{m+1}, \nabla \mathbf{v}_h) - (p_h^m, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h)$$

- (b) Compute $\Pi_h(\nabla \cdot \mathbf{u}_h^{m+1})$.

- (c) Find $p_h^{m+1} \in Q_h$ such that

$$(\nabla(p_h^{m+1} - p_h^m + \nu \Pi_h(\nabla \cdot \mathbf{u}_h^{m+1})), \nabla q_h) = \left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{k}, \nabla q_h \right) \quad \forall q_h \in Q_h.$$

- The *penalty pressure-projection* scheme:

k	0.2 – 0.1	0.1 – 0.05	0.05- 0.025
$\ u_1\ _{l^\infty(L^2)}$	1.077	1.326	1.582
$\ u_1\ _{l^\infty(H^1)}$	0.812	1.146	1.453
$\ u_2\ _{l^\infty(L^2)}$	1.095	1.352	1.585
$\ u_2\ _{l^\infty(H^1)}$	0.817	1.148	1.457
$\ p\ _{l^2(L^2)}$	0.877	1.282	1.535
$\ p\ _{l^\infty(L^2)}$	0.880	1.157	1.444

Table 1: Error orders in time for Algorithm 2

k	0.2 – 0.1	0.1 – 0.05	0.05- 0.025
$\ u_1\ _{l^\infty(L^2)}$	1.048	1.278	1.475
$\ u_1\ _{l^\infty(H^1)}$	0.955	1.150	1.290
$\ u_2\ _{l^\infty(L^2)}$	1.105	1.314	1.511
$\ u_2\ _{l^\infty(H^1)}$	1.035	1.176	1.311
$\ p\ _{l^2(L^2)}$	1.241	1.436	1.490
$\ p\ _{l^\infty(L^2)}$	1.012	1.238	1.361

Table 2: Error orders in time for Rotational Scheme

(a) Find $\mathbf{u}_h^{m+1} \in \mathbf{Y}_h$ such that, $\forall \mathbf{v}_h \in \mathbf{Y}_h$,

$$\left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{k}, \mathbf{v}_h \right) + c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1}, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^{m+1}, \nabla \mathbf{v}_h) + \nu(\nabla \cdot \mathbf{u}_h^{m+1}, \nabla \cdot \mathbf{v}_h) - (q_h^m, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h).$$

where q_h^m is given as in (47). Note that this scheme is not fully segregated because it couples the velocity components in the term $\nu(\nabla \cdot \mathbf{u}_h^{m+1}, \nabla \cdot \mathbf{v}_h)$.

(b) Compute $\Pi_h(\nabla \cdot \mathbf{u}_h^{m+1})$.

(c) Find $p^{m+1} \in Q_h$ solving the Poisson-Neumann problem (48).

We consider the structured mesh taking 70 subintervals in $[0, 1]$ (with $h = 0.0142857$). In addition, $k = 0.2, 0.1, 0.05$ and 0.025 are considered corresponding to 10, 20, 40 and 80 time iterations in the time interval $[0, 2]$.

The numerical results comparing the time accuracy can be seen in Tables 1, 2, 3 and 4, showing a little better accuracy in velocity and pressure for the incremental scheme Algorithm 2. Moreover, first order accurate in time for velocity and pressure is observed for all previous schemes.

With respect to the computational cost, the CPU time needed taking $k = 0.025$ (80 time iterations) is shown in Table 5, showing a little lower cost in the incremental scheme Algorithm 2.

k	$0.2 - 0.1$	$0.1 - 0.05$	$0.05 - 0.025$
$\ u_1\ _{l^\infty(L^2)}$	0.726	0.814	0.885
$\ u_1\ _{l^\infty(H^1)}$	0.715	0.813	0.885
$\ u_2\ _{l^\infty(L^2)}$	0.764	0.843	0.905
$\ u_2\ _{l^\infty(H^1)}$	0.775	0.841	0.908
$\ p\ _{l^2(L^2)}$	0.822	0.906	0.952
$\ p\ _{l^\infty(L^2)}$	0.700	0.792	0.868

Table 3: Error orders in time for Consistent Scheme

k	$0.2 - 0.1$	$0.1 - 0.05$	$0.05 - 0.025$
$\ u_1\ _{l^\infty(L^2)}$	0.983	1.256	1.459
$\ u_1\ _{l^\infty(H^1)}$	0.903	1.120	1.266
$\ u_2\ _{l^\infty(L^2)}$	1.012	1.265	1.484
$\ u_2\ _{l^\infty(H^1)}$	0.942	1.135	1.282
$\ p\ _{l^2(L^2)}$	1.161	1.354	1.429
$\ p\ _{l^\infty(L^2)}$	0.937	1.172	1.324

Table 4: Error orders in time for Penalty-Projection Scheme

Note that in this scheme the problem related to the $L^2(\Omega)$ -projector Π_h has not to be computed.

4 Conclusions

The optimal error estimates of order $O(k + h)$ for the velocity and pressure are deduced for the first-order linear fully discrete segregated scheme based on an incremental pressure projection method (Algorithm 2) approaching the 3D Navier-Stokes problem. This convergence is unconditional, i.e. without imposing constraints on mesh size h or time step k .

Moreover, some numerical computations of the segregated version of Algorithm 2 agree the previous numerical analysis are provided. These simulations are also compared with the segregated versions of the rotational, consistent and penalty-projection schemes, obtaining a little better accuracy in time and lower computational cost of Algorithm 2.

Finally, although this segregated scheme has the numerical boundary layer furnished by the

Scheme:	Algorithm 2	Rotational	Consistent	Penalty
CPU-time (s)	2067.45	2113.22	2079.4	2147.9

Table 5: Computational cost

artificial boundary condition $\nabla(p^{m+1} - p^m) \cdot \mathbf{n}$ on $\partial\Omega$, this fact does not perturb the optimal convergence in the energy norms $\mathbf{H}^1(\Omega) \times L^2(\Omega)$ for the velocity and pressure, respectively.

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